Proving Continuity of Coinductive Global Bisimulation Distances: A Never Ending Story*

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We have developed a global approach to define bisimulation distances which goes somehow further away than the bisimulation distances based on the bisimulation game, previously proposed by some other authors. Our proposal is based on the cost of transformations: how much we need to modify one of the compared processes to obtain the other. Our original definition only covered finite processes, but a coinductive approach extends it to cover infinite but finitary trees. We have shown many interesting properties of our distances, and we wanted to prove their continuity with respect to projections, bur unfortunately we have not been able to accomplish that task. However, we have obtained several partial results that we now present in this paper.

1 Introduction

The notion of bisimulation has been extensively used to characterize the equivalence between processes [11, 12, 17]. Bisimulations are coinductive proofs of that equivalence, which is called the bisimilarity relation. Certainly, bisimilarity is a quite natural relation, the many different ways that we have to prove and disprove bisimilarity somehow indicates it . The bisimulation game [18] makes both things as the same time. Moreover, it gives us unordered trees without repeated (equivalent) branches, as the canonical model for the semantics.

Therefore, two process are equivalent iff they have the same tree as semantics. But when two processes are not equivalent we have no way of expressing "how different" they are. Recently, some authors have defined several notions of bisimulation distance by means of variants of the bisimulation game [1, 4, 6, 7]: while the original game imposes to the defender the obligation of (exactly) replicating any move by the attacker, in these variants the defender has the possibility of "cheating", by replying an attacker's move by choosing similar, but not equal, actions. However, when doing that, the defender have to pay a price according to the distance between the two involved actions.

This is a very suggestive path to follow when defining a bisimulation distance. Indeed, the generalized bisimulation provides a simple and natural way of describing the distance between processes, and offers several efficient ways to compute it. However, despite such desirable properties, we believe that alternative approaches exist, that are worth being studied. The two first authors of this paper have presented in their previous works on the subject [14, 15, 16] several examples which mainly show that the

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"classical" distances instead of being "additive" (aggregating all the differences between the processes) are "local", looking for the biggest (concrete) difference, and thus (totally) disregarding all the rest.

Our approach [14, 15, 16] tried to correct this situation: since trees express the bisimulation semantics, we looked for a natural distance between trees that defines what we called our global bisimulation distance: it tries to aggregate all the "reasons" why two processes are not equivalent. This was done by introducing a simple rule to capture "punctual" differences between processes. Then we defined the sequences of transformations, each one of them providing a bound for the distance between the two connected processes.

This finite way of defining bounds for the distances is useful when comparing finite processes, but it is clearly inadequate when comparing infinite processes with infinitely many differences between them. We are interested in obtaining sound bounds also in this case, whenever the series collecting all those differences converges. Instead of looking for a complex scenario based on the use of limits, we introduced in [16] a coinductive framework which obtains the bounds for those distances in a very simple way (coinduction is sometimes presented as an "inductionless induction" mechanism).

As we said before, classical bisimulation distances are easy to calculate, even (or we should better say, specially) in the quantitative cases (e.g., probabilistic [20], timed [19]), where calculus provides the machinery to obtain the corresponding fixed points. It is true that the computation of our (bounds for the) distances requires specific techniques in each case, but our coinductive approach benefits from the power of coinduction to accomplish this task.

In order to give a broader support to our approach, we wanted to prove the continuity of our distances: we expected that whenever all the pairs of projections of two (possibly infinite) processes are at some fixed distance, the (full) processes themselves will be at that distance. Unfortunately, this paper tells an unfinished story: even if we expected a perhaps bit involved, but somehow "standard" proof of completeness, our creature has revealed itself as an irresistible beast: our, more and more, sophisticated attempts to domesticate it have crashed over and over, revealing possible faces that perhaps do not really exist. However, we all know how difficult is to disprove the existence of Nessy, Bigfoot, or E.T. Certainly, any counterexample meant to capture and show any of these strange creatures will definitely terminate with the mystery, but still we believe that this will be impossible, since they simply do not exist …or they do?

Let us go with our story. It is not easy to tell unfinished stories, but we think that we have to do it, since we really enjoyed many exciting adventures along our search, and perhaps our (practical) discoveries will help some others to finish the work, or at least it will avoid that they again become lost where we were.

2 Classical and global bisimulation distances

Our starting point will be the operational definition of processes defined by *Labelled Transition Systems (lts)*, which is given by a set of states N, and a function $succ : N \to \mathscr{P}(\mathbb{A} \times N)$. We denote them by $(N, succ)$, when we want to distinguish an initial state n_0, we add it to get $(N, succ, n_0)$. Sometimes we will remove the *succ* component, simply considering (N, n_0). Finite computations or *paths*, are sequences $n_0 a_1 n_1 \dots a_k n_k$ with $(a_{i+1}, n_{i+1}) \in succ(n_i)\ \forall i \in \{0 \dots k-1\}$. We denote the set of paths by $Path(N, n_0)$.

Definition 1. *We say that an lts* (N, n_0) *is (or defines) a* tree t *if for all* $n \in N$ *there is a single path* $n_0 a_1 n_1 \dots a_j n_j$ *with* $n_j = n$. *Then, we say that each node* n_k *is at* level k *in* t, *and define* $Level_k(t) = \{n \in N \mid n \text{ is at level } k \text{ in } t\}$. *We define the* depth *of* t *as* $depth(t) = sup\{l \in \mathbb{N} \mid Level_l(t) \neq \emptyset\} \in \mathbb{N} \cup \{\infty\}$. *We denote by* $Trees(\mathbb{A})$ *the class of trees on the set* $\mathbb{A}$, *and by* $FTrees(\mathbb{A})$, *the subclass of finite state trees.*

Every node $n \in N$ of a tree $t = (N, n_0)$ induces a subtree $t_n = (N_n, n)$, where N_n is the set of nodes $n'_k \in N$ such that there exists a path $n'_0 a_1 n'_1 \dots a_k n'_k$ with $n'_0 = n$. We decompose any tree t into the formal sum $\sum_{n_{1j} \in Level_1(t)} a_j t_{n_{1j}}$. Since our trees are unordered, by definition, this formal sum is also unordered.

For each tree $t \in Trees(\mathbb{A})$, we define its *first-level width*, that we represent by $||t||$, as $||t|| = |Level_1(t)|$. We also define the *first k-levels width of t*, denoted by $||t||_k$, as $||t||_k = max\{||t_n|| \mid n \in \bigcup_{l \le k} Level_l(t)\}$. *Finitary trees* are those such that $||t||_k < \infty$, $\forall k \in \mathbb{N}$. We denote by $FyTrees(\mathbb{A})$ the collection of *finitary trees* in $Trees(\mathbb{A})$.

Definition 2. *Given an lts with initial state $(N, succ, n_0)$, we define its unfolding, unfold(N), as the tree $(\overline{N}, \overline{succ}, \overline{n_0})$, where $\overline{N} = Path(N, n_0)$, $\overline{succ}(n_0 a_1 \dots n_k) = \{(a, n_0 a_1 \dots n_k a n') \mid (a, n') \in succ(n_k)\}$, and $\overline{n_0} = n_0$. An lts is finitely branching when its unfolding is a finitary tree.*

Definition 3. *Given a tree $t = (N, succ, n_0)$ and $k \in \mathbb{N}$, we define its k-th cut or projection, $\pi_k(t)$, as the restriction of t to the nodes in $\bigcup_{l \le k} Level_l(t)$:*

$$\pi_k(t) = (\pi_k(N), succ_k, n_0), \text{ where } \pi_k(N) = \bigcup_{l \le k} Level_l(t),\ succ_k(n) = succ(n) \text{ for } n \in \bigcup_{l < k} Level_l(t), \text{ and } succ_k(n) = \emptyset \text{ if } n \in Level_k(t).$$

Each finitary tree is unequivocally defined by its sequence of projections: $\forall t, t' \in FyTree(\mathbb{A})((\forall k \in \mathbb{N}\, \pi_k(t) = \pi_k(t')) \Rightarrow t = t')$. We focus on finitely branching lts, and thus on finitary trees.

We consider domains of actions $(\mathbb{A}, \mathbf{d})$, where $\mathbf{d} : \mathbb{A} \times \mathbb{A} \to \mathbb{R}^+ \cup \{\infty\}$ is a distance between actions, with $\mathbf{d}(a,b) = \mathbf{d}(b,a)$, $\mathbf{d}(a,b) = 0 \Leftrightarrow a = b$, and $\mathbf{d}(a,c) + \mathbf{d}(c,b) \ge \mathbf{d}(a,b)$, $\forall a, b, c \in \mathbb{A}$, where $+$ is extended to $\mathbb{R}^+ \cup \{\infty\}$ as usual. We assume that $\mathbf{d}(a,b) = \infty$ when the value $\mathbf{d}(a,b)$ is not specified.

When comparing pairs of processes, it is natural [2, 7, 3] to introduce a "discount factor" $\alpha \in (0,1]$. Then, the differences in the k-th level of the compared trees are weighted by α^k, following the idea that differences in the far future are less important than those in the near. As a consequence, it is possible to obtain finite distances when comparing two processes with "infinitely many differences" between them.

In [14], we have presented our operational definitions that allow us to obtain bounds for our *global distances* between finite trees. These bounds are given by the cost of any transformation that turns one of the trees into the other. The following definition states which are the valid steps of those transformations and their costs. Roughly, any application of idempotency of $+$ has no cost, while the change of an action a at level k into another b, costs $\alpha^k \mathbf{d}(b,a)$.

Definition 4. *Given a domain of actions $(\mathbb{A}, \mathbf{d})$ and a discount factor $\alpha \in (0,1]$, we inductively define the distance steps on $FTrees(\mathbb{A})$ by*

1. $d \ge 0 \Rightarrow (t \rightsquigarrow^1_{\alpha,d} t + t \wedge t + t \rightsquigarrow^1_{\alpha,d} t)$.
2. $d \ge \mathbf{d}(a,b) \Rightarrow at \rightsquigarrow^1_{\alpha,d} bt$.
3. $t \rightsquigarrow^1_{\alpha,d} t' \Rightarrow t + t'' \rightsquigarrow^1_{\alpha,d} t' + t''$.
4. $t \rightsquigarrow^1_{\alpha,d} t' \Rightarrow at \rightsquigarrow^1_{\alpha,\alpha d} at'$.

We associate to each distance step its level. The level of any step generated by 1. or 2. is one; while if the level of the corresponding premise $t \rightsquigarrow^1_{\alpha,d} t'$ is k, then the level of a step generated by 3. (resp. 4.) is k (resp. $k+1$). Finally, we define the family of global distance relations $\langle \rightsquigarrow_{\alpha,d} \mid d \in \mathbb{R}^+ \rangle$, taking $t \rightsquigarrow_{\alpha,d} t'$ if there exists a sequence $\mathscr{S} := t = t^0 \rightsquigarrow^1_{\alpha,d_1} t^1 \rightsquigarrow^1_{\alpha,d_2} t^2 \rightsquigarrow^1_{\alpha,d_3} \cdots \rightsquigarrow^1_{\alpha,d_n} t^n = t'$, with $\sum_{i=1}^n d_i \le d$.

3 The coinductive global bisimulation distance

To extend our global distances to infinite trees, we have introduced in [16] a general coinductive notion of distance. We formalize our definition in two steps. In the first one we introduce the rules that produce the steps of the *coinductive transformations* between trees, starting from any family of triples (t, t', d), with $t, t' \in FyTrees(\mathbb{A})$ and $d \in \mathbb{R}^+$.

Definition 5 ([16]). *Given a domain of actions $(\mathbb{A},\mathbf{d})$, a discount factor $\alpha \in (0,1]$ and a family $\mathscr{D} \subseteq FyTrees(\mathbb{A}) \times FyTrees(\mathbb{A}) \times \mathbb{R}^+$, we define the family of relations $\equiv_d^{\mathscr{D},\alpha}$, by:*

1. *For all $d \geq 0$ we have (i) $(\sum_{j\in J} a_j t_j) + at + at \equiv_d^{\mathscr{D},\alpha} (\sum_{j\in J} a_j t_j) + at$,*
   and (ii) $(\sum_{j\in J} a_j t_j) + at \equiv_d^{\mathscr{D},\alpha} (\sum_{j\in J} a_j t_j) + at + at$.
2. *For all $d \geq \mathbf{d}(a,b)$ we have $(\sum_{j\in J} a_j t_j) + at \equiv_d^{\mathscr{D},\alpha} (\sum_{j\in J} a_j t_j) + bt$.*
3. *For all $(t,t',d) \in \mathscr{D}$ we have $(\sum_{j\in J} a_j t_j) + at \equiv_{d'}^{\mathscr{D},\alpha} (\sum_{j\in J} a_j t_j) + at'$ for all $d' \geq \alpha d$.*

To simplify the notation, we write $\equiv_d$ instead of $\equiv_d^{\mathscr{D},\alpha}$, whenever $\mathscr{D}$ and α are clear from the context.

Inspired by the proof obligations imposed to bisimulations we introduce the coinductive proof obligations imposed to the (satisfactory) families of distances.

Definition 6 ([16]). *Given a domain of actions $(\mathbb{A},\mathbf{d})$ and a discount factor $\alpha \in (0,1]$, we say that a family $\mathscr{D}$ is an α-coinductive collection of distances (α-ccd) between finitary trees, if for all $(t,t',d) \in \mathscr{D}$ there exists a* finite coinductive transformation sequence *$\mathscr{C} := t = t^0 \equiv_{d_1} t^1 \equiv_{d_2} \ldots \equiv_{d_n} t^n = t'$, with $d \geq \sum_{j=1}^n d_j$. Then, when there exists an α-ccd $\mathscr{D}$ with $(t,t',d) \in \mathscr{D}$, we will write $t \equiv_d^\alpha t'$, and say that tree t is at most at distance d from tree t' wrt α.*

We say that the steps generated by application of rules 1 and 2 in Def. 5 are *first level steps*; while those generated by rule 3 are *coinductive steps*.

Example 1. *Let us consider the domain of actions $(\mathbb{N},\mathbf{d})$, where $\mathbf{d}$ is the usual distance for numbers, and the trees $t_N = unfold(N)$ and $t_{N'} = unfold(N')$, with $N = \{n_0,n_1\}$, $succ(n_0) = \{(0,n_0),(0,n_1)\}$ and $succ(n_1) = \emptyset$; and $N' = \{n'_0,n'_1\}$, $succ'(n'_0) = \{(0,n'_0),(1,n'_1)\}$ and $succ'(n'_1) = \emptyset$. Then, we have $t_N \equiv_2^{\mathscr{D},1/2} t_{N'}$, using the family $\mathscr{D} = \{(t_N,t_{N'},2)\}$. We can prove that this is indeed a $\frac{1}{2}$-ccd, by considering the sequence: $\mathscr{C} := t_N \equiv_1^{\mathscr{D},1/2} t_{N''} \equiv_1^{\mathscr{D},1/2} t_{N'}$, where $t_{N''} = unfold(N'')$, with $N'' = \{n''_0,n''_1,n''_2,n''_3\}$, $succ''(n''_0) = \{(1,n''_1),(0,n''_2)\}$, $succ''(n''_1) = \emptyset$, $succ''(n''_2) = \{(0,n''_2),(0,n''_3)\}$ and $succ''(n''_3) = \emptyset$.*

It is immediate to see that our notion of distance has natural and pleasant properties such as the *triangular transitivity*: for any discount factor $\alpha \in (0,1]$, whenever we have $t \equiv_d^\alpha t'$ and $t' \equiv_{d'}^\alpha t''$, we also have $t \equiv_{d+d'}^\alpha t''$. Of course, our coinductive definition of the global bisimulation distance generalizes our operational definition for finite trees.

Proposition 1 ([16]). *For $t,t' \in FTrees(\mathbb{A})$, the operational (Def. 4) and the coinductive (Def. 6) definitions of our distance between trees coincide, that means $t \equiv_d^\alpha t' \Leftrightarrow t \rightsquigarrow_{\alpha,d} t'$.*

Proof. $\Rightarrow|$ Given an α-ccd relating finite trees with $(t,t',d) \in \mathscr{D}$, we can "unfold" the corresponding sequence, $\mathscr{C}$, witnessing $t \equiv_d^\alpha t'$, into a sequence of distance steps, $\mathscr{S}$, proving that $t \rightsquigarrow_{\alpha,d} t'$. We proceed by induction on $depth(t)$: Let $t^i \equiv_{d_i}^{\mathscr{D},\alpha} t^{i+1}$ be an intermediate step in the coinductive sequence $\mathscr{C}$. There are no steps with $depth(t) = 0$, so that the base case is trivial. For $depth(t) \geq 1$, if $t^i \equiv t^{i+1}$ has been obtained by applying rule 2 in Def. 5, then we have $t^i = (\sum_{j\in J} a_j t_j^i) + at \equiv_{d_i} (\sum_{j\in J} a_j t_j^i) + bt = t^{i+1}$, with $d_i \geq \alpha\mathbf{d}(a,b)$ and applying rules 2 and 3 in Def. 4 we also obtain $t^i \rightsquigarrow_{\alpha,d_i} t^{i+1}$. If the steps was obtained by applying rule 3 in Def. 5, getting $t^i = t_1^i + at_1 \equiv_{d_i}^{\mathscr{D},\alpha} t_1^i + at'_1 = t^{i+1}$ for $(t_1,t'_1,d_i/\alpha) \in \mathscr{D}$, we apply the induction hypothesis, getting $t_1 \rightsquigarrow_{\alpha,d_i/\alpha} t'_1$, and using rules 4 and 3 in Def. 4, we obtain the goal $t^i = t_1^i + at_1 \rightsquigarrow_{\alpha,d_i} t_1^i + at'_1 = t^{i+1}$.

$\Leftarrow|$ Given a sequence of distance steps, $\mathscr{S}$, proving that $t \rightsquigarrow_{\alpha,d} t'$, we can "fold" it into a coinductive sequence, $\mathscr{C}$, witnessing $t \equiv_d^\alpha t'$. For each $(t,t',d) \in \mathscr{D}$ we consider the factorization of the sequence $\mathscr{S}$ and its reordering as done in [16]. We get $t = \sum_{i\in I_0} a_i t_i \rightsquigarrow_{\alpha,d_{02}} \sum_{i\in I_0} a_i t'_i \rightsquigarrow_{\alpha,d_{11}}^1 \sum_{i\in I_1} a_i t_i \rightsquigarrow_{\alpha,d_{12}} \sum_{i\in I_1} a_i t'_i \rightsquigarrow_{\alpha,d_{21}}^1 \cdots \rightsquigarrow_{\alpha,d_{(k+1)2}} \sum_{i\in I_{k+1}} a_i t'_i = t'$, where for each sequence $\sum_{i\in I_j} a_i t_i \rightsquigarrow_{\alpha,d_{j2}} \sum_{i\in I_j} a_i t'_i$ and each $i \in I_j$, we have $t_i \rightsquigarrow_{\alpha, d^i_{j2}/\alpha} t'_i$ with $\sum_{i\in I_j} d^i_{j2} = d_{j2}$. Now, applying the induction hypothesis, we have $(t_i, t'_i, d^i_{j2}/\alpha) \in \mathscr{D}$, for all $i \in I_j$, so that $\sum a_i t_i \equiv^{\mathscr{D},\alpha}_{\alpha d_1} \sum a_i t^1_i \equiv^{\mathscr{D},\alpha}_{\alpha d_2} \sum a_i t^2_i \equiv^{\mathscr{D},\alpha}_{\alpha d_3} \cdots \equiv^{\mathscr{D},\alpha}_{\alpha d_{|I_j|}} \sum a_i t^{|I_j|}_i = \sum a_i t'_i$.

Therefore, each sequence $\sum_{i\in I_j} a_i t_i \rightsquigarrow_{\alpha, d^i_{j2}} \sum_{i\in I_j} a_i t'_i$ at the factorization above can be substituted by a sequence of $|I_j|$ valid coinductive steps, getting a total distance $\sum_{j=0}^{k+1} \sum_{i\in I_j} d^i_{j1} + \sum_{j=0}^{k+1} \sum_{k=1}^{|I_j|} \alpha d_k = d$. □

Even if the result above only concerns finite trees, it reveals the duality between induction and coinduction. Of course, its consequences are much more interesting in the infinite case.

Example 2. *Let $(\{a,b\}, \mathbf{d})$ be a domain of actions such that $\mathbf{d}(a,b) = 1$ and let us consider the lts given by $N_{1,\infty} = \{n_0\}$, with $succ(n_0) = \{(a, n_0)\}$ and its unfolding a^∞. In an analogous way, we obtain the tree b^∞. We have $\pi_n(a^\infty) = a^n$, and clearly a^∞ can be seen as the "limit" of its projections. Applying Def. 4, we obtain $a^n \rightsquigarrow_{1/2,2} b^n$ and thus $a^n \equiv^{1/2}_2 b^n$. We also have $a^\infty \equiv^{1/2}_2 b^\infty$, which can indeed be proved by means of the (trivial!) collection $\mathscr{D} = \{(a^\infty, b^\infty, 2)\}$. We can check that $\mathscr{D}$ is an $\frac{1}{2}$-ccd using the coinductive sequence $\mathscr{C} := a^\infty = aa^\infty \equiv^{\mathscr{D},1/2}_1 ba^\infty \equiv^{\mathscr{D},1/2}_{\frac{1}{2}\cdot 2} bb^\infty = b^\infty$.*

4 On the continuity of the global bisimulations distance

It is very simple to prove the following proposition, by introducing the notion of projections of α-ccd's.

Proposition 2 ([16]). *For any α-ccd $\mathscr{D}$, the projected family $\pi(\mathscr{D}) = \{(\pi_n(t), \pi_n(t'), d) \mid (t, t', d) \in \mathscr{D},\ n \in \mathbb{N}\}$ is an α-ccd that proves $t \equiv^\alpha_d t' \Rightarrow \forall n \in \mathbb{N}\ \pi_n(t) \equiv^\alpha_d \pi_n(t')$.*

Proof. Let $\mathscr{C} := t = t^0 \equiv^{\mathscr{D},\alpha}_{d_1} \ldots \equiv^{\mathscr{D},\alpha}_{d_k} t^k = t'$ be the coinductive sequence proving that $(t, t', d) \in \mathscr{D}$ satisfies the condition in order $\mathscr{D}$ to be an α-ccd. Then each projected sequence $\pi_n(\mathscr{C}) := \pi_n(t) = \pi_n(t^0) \equiv^{\pi(\mathscr{D}),\alpha}_{d_1} \ldots \equiv^{\pi(\mathscr{D}),\alpha}_{d_k} \pi_n(t^k) = \pi_n(t')$ proves that $(\pi_n(t), \pi_n(t'), d) \in \pi(\mathscr{D})$ satisfies the condition in order $\pi(\mathscr{D})$ to be an α-ccd. It is clear that the projection under π_n of any first level step in $\mathscr{C}$, is also a valid step in $\pi_n(\mathscr{C})$. Moreover, any coinductive step in $\mathscr{C}$ using $(t_1, t'_1, d) \in \mathscr{D}$, can be substituted by the corresponding projected step, that uses $(\pi_{n-1}(t_1), \pi_{n-1}(t'_1), d) \in \pi(\mathscr{D})$. □

Remark 1. *Alternatively, we can consider for each $n \in \mathbb{N}$ a family $\mathscr{D}_n = \pi_n(\mathscr{D}) = \{(\pi_m(t), \pi_m(t'), d) \mid (t, t', d) \in \mathscr{D},\ m \in \mathbb{N}, m \le n\}$, using the fact that the subtrees of a projection $\pi_n(t)$ are also projections $\pi_m(t'')$ of subtrees t'' of t, for some $m < n$. These families satisfy $\pi_m(\mathscr{D}) \subseteq \pi_n(\mathscr{D})$, whenever $m \le n$.*

We still conjecture that the converse of Prop. 2 asserting the continuity of our coinductive distance, is also valid. But, is is also true, that once repeated tries have failed, or conduced us into a blind alley, we are now less sure of that result than in the beginning. We expected to use the reasoning in the proof of Prop. 2 in the reversed direction. If we would have a collection of "uniform"[1] operational sequences $\mathscr{S}^n := \pi_n(t) \rightsquigarrow_{\alpha,d} \pi_n(t')$, we could "overlap" all of them getting an "infinite tree-structured sequence". By "folding" it we would obtain the coinductive sequence proving that $t \equiv^\alpha_d t'$. Next, a simple example.

Example 3 ([16]). *Let us consider the trees $t = ac^\infty + ad^\infty$ and $t' = bc^\infty + bd^\infty$, and the distance $\mathbf{d}$ defined by $\mathbf{d}(a,b) = 4$, $\mathbf{d}(c,d) = 1$. Then we have:*

[1] Uniformity here means that for any $n, k \in \mathbb{N}$, with $k \ge n$, the steps of all the sequences $\mathscr{S}^k$ corresponding to the first n-levels are always the same.

$$\begin{array}{ccc}
\mathscr{C}^n & \longrightarrow & unfold(\mathscr{C}^n) := \mathscr{S}^n \\
\pi^m \big\downarrow & & \big\downarrow \pi^m \\
\mathscr{C}^m & \longrightarrow & unfold(\mathscr{C}^m) := \mathscr{S}^m
\end{array}$$

Figure 1: Relating projections of sequences

$$\pi_1(t) = a + a \rightsquigarrow^{(1)}_{\frac{1}{2},0} a \rightsquigarrow^{(1)}_{\frac{1}{2},4} b \rightsquigarrow^{(1)}_{\frac{1}{2},0} b + b = \pi_1(t'),$$
$$\pi_2(t) = ac + ad \rightsquigarrow^{(2)}_{\frac{1}{2},\frac{1}{2}\cdot 1} ac + ac \rightsquigarrow^{(1)}_{\frac{1}{2},0} ac \rightsquigarrow^{(1)}_{\frac{1}{2},4} bc \rightsquigarrow^{(1)}_{\frac{1}{2},0} bc + bc \rightsquigarrow^{(2)}_{\frac{1}{2},\frac{1}{2}\cdot 1} bc + bd = \pi_2(t'),$$
$$\pi_3(t) = acc + add \rightsquigarrow^{(2)}_{\frac{1}{2},\frac{1}{2}\cdot 1} acc + acd \rightsquigarrow^{(3)}_{\frac{1}{2},\frac{1}{4}\cdot 1} acc + acc \rightsquigarrow^{(1)}_{\frac{1}{2},0} acc \rightsquigarrow^{(1)}_{\frac{1}{2},4}$$
$$bcc \rightsquigarrow^{(1)}_{\frac{1}{2},0} bcc + bcc \rightsquigarrow^{(2)}_{\frac{1}{2},\frac{1}{2}\cdot 1} bcc + bdc \rightsquigarrow^{(3)}_{\frac{1}{2},\frac{1}{4}\cdot 1} bcc + bdd = \pi_3(t').$$

Now, if we consider the operational sequences, $\mathscr{S}^n$, relating $\pi_n(t)$ and $\pi_n(t')$, for any $n \in \mathbb{N}$, we obtain $\pi_n(t) \rightsquigarrow_{\frac{1}{2},d_n} \pi_n(t')$, for some $d_n < 6$. Applying Prop. 1 we can turn these operational sequences into equivalent coinductive ones, $\mathscr{C}^n$, thus proving $\pi_n(t) \equiv_6^{1/2} \pi_n(t')$, for all $n \in \mathbb{N}$:

$$\mathscr{C}^1 := a + a \equiv_0 a \equiv_4 b \equiv_0 b + b,$$
$$\mathscr{C}^2 := ac + ad \equiv_{\frac{1}{2}} ac + ac \equiv_0 ac \equiv_4 bc \equiv_0 bc + bc \equiv_{\frac{1}{2}} bc + bd,$$
$$\mathscr{C}^3 := acc + add \equiv_{\frac{1}{2}} acc + acd \equiv_{\frac{1}{4}} acc + acc \equiv_0 acc \equiv_4 bcc \equiv_0 bcc + bcc \equiv_{\frac{1}{2}} bcc + bdc \equiv_{\frac{1}{4}} bcc + bdd.$$

The continuity of our coinductive distances would mean that, whenever we have $\pi_n(t) \equiv_d^\alpha \pi_n(t')$ $\forall n \in \mathbb{N}$, there should be a collection of "uniform" sequences proving these facts. Certainly, this is the case when we know in advance that $t \equiv_d^\alpha t'$. Next, we define the projection of our operational sequences.

Lemma 1. *If $t \rightsquigarrow^1_{\alpha,d} t'$ is a distance step of level l, then $\pi_k(t) \rightsquigarrow^1_{\alpha,d} \pi_k(t')$ is also a distance step of level l for all $k \geq l$. Instead, if $l > k$ then we have $\pi_k(t) = \pi_k(t')$.*

Definition 7. *Given a sequence of distance steps $\mathscr{S} := t = t^0 \rightsquigarrow^1_{\alpha,d_1} t^1 \rightsquigarrow^1_{\alpha,d_2} \cdots \rightsquigarrow^1_{\alpha,d_n} t^n = t'$ with $d = \sum_{i=1}^n d_i$, for any $k \in \mathbb{N}$ we define the sequence $\pi_k(\mathscr{S}) := \pi_k(t) = \pi_k(t^0) \rightsquigarrow^1_{\alpha,d_{i_1}} \pi_k(t^{i_1}) \rightsquigarrow^1_{\alpha,d_{i_2}} \cdots \rightsquigarrow^1_{\alpha,d_{i_l}}$ $\pi_k(t^{i_l}) = \pi_k(t')$, where $\langle i_1, i_2, \ldots, i_l \rangle$ is the sequence of indexes i_j for which the level of the distance step $t^{i_j - 1} \rightsquigarrow^1_{\alpha,d_{i_j}} t^{i_j}$ is less or equal than k, while the rest of the steps are removed.*

Proposition 3. *For any $k \in \mathbb{N}$, and any sequence of distance steps $\mathscr{S}$, the projected sequence $\pi_k(\mathscr{S})$ is a sequence of distance steps, thus proving $\pi_k(t) \rightsquigarrow_{\alpha,\sum d_{i_j}} \pi_k(t')$. Therefore, we also have $\pi_k(t) \rightsquigarrow_{\alpha,d} \pi_k(t')$.*

Now, let us start with a coinductive sequence $\mathscr{C}$ relating two (possibly infinite) trees t and t'.

Corollary 1. *The projection of coinductive sequences and those of distance steps that the former induces, are related by the commutative diagram in Fig. 1. There we denote by $\mathscr{C}^n$ the projections of a coinductive sequence $\mathscr{C}$.*

Note that all these coinductive sequences have exactly the "same structure", which is formalized by the fact that $\pi_m(\mathscr{C}^n) = \mathscr{C}^m$, for all $m \leq n$. As a consequence, if now we forget that we have already the original family $\mathscr{D}$, starting from the families $\mathscr{D}_n$ in Remark 1, we could "reverse" the procedure by means of which they were defined, obtaining a single family $\mathscr{D}' = \bigcup \mathscr{D}_n \cup \{(a^\infty, b^\infty, 2)\} = \{(a^k, b^k, 2) \mid k \leq \infty\}$, where we have added the "limit" triple $(a^\infty, b^\infty, 2)$ because for all $k \in \mathbb{N}$ $(\pi_k(a^\infty), \pi_k(b^\infty), 2) = (a^k, b^k, 2) \in \bigcup \mathscr{D}_n$. Now, we can see that $\mathscr{D}'$ is indeed a $\frac{1}{2}$-ccd. In order to check the condition corresponding to

$(a^\infty, b^\infty, 2)$, we "overlap" the sequences $\mathscr{C}^n$ getting a sequence $\mathscr{C}'$ constructed as follows: From the first level step $a^n = aa^{n-1} \equiv_1^{\mathscr{D}_n,1/2} ba^{n-1}$ at each $\mathscr{C}^n$, we obtain the first level step $a^\infty = aa^\infty \equiv_1^{\mathscr{D}',1/2} ba^\infty$; this is followed by the coinductive step $ba^\infty \equiv_{\frac{1}{2}\cdot 2}^{\mathscr{D}',1/2} bb^\infty = b^\infty$, obtained just removing the projections from any of the coinductive steps $ba^{n-1} = \pi_n(ba^\infty) \equiv_{\frac{1}{2}\cdot 2}^{\mathscr{D}_n,1/2} \pi_n(bb^\infty) = \pi_n(b^\infty) = bb^{n-1} = b^n$.

The important fact about the construction above is that it can be applied to any collection of coinductive sequences that "match each other". We will call these collections telescopic, because when we "unfold" their elements we obtain a sequence of operational sequences, where any of them is obtained from the previous one by adding some new intermediate steps, that always correspond to transformation steps at the last level of the compared trees (this reminds us the "opening" of a telescopic antenna).

Definition 8. *Let $t, t' \in FyTrees(\mathbb{A})$ and let $(\mathscr{S}^n)_{n\in\mathbb{N}}$ be a collection of operational sequences proving $\pi_n(t) \rightsquigarrow_{\alpha,d} \pi_n(t')$. We say that it is* telescopic[2] *if $\pi_m(\mathscr{S}^n) = \mathscr{S}^m$, for all $m \le n$.*

Any telescopic collection $(\mathscr{S}^n)_{n\in\mathbb{N}}$, produces a "limit" coinductive sequence $\mathscr{C}$ proving $t \equiv_d^\alpha t'$:

Lemma 2. *Let $(\mathscr{S}^n)_{n\in\mathbb{N}}$ be a telescopic collection relating t and t'. We consider the associated factorization of each sequence in it: $\mathscr{S}^n := \pi_n(t) = t^{n,0,1} \rightsquigarrow_{\alpha,d_{n02}} t^{n,0,2} \rightsquigarrow^1_{\alpha,d_{n11}} t^{n,1,1} \cdots \rightsquigarrow^1_{\alpha,d_{nk1}} t^{n,k,1} \rightsquigarrow_{\alpha,d_{nk2}} t^{n,k,2} = \pi_n(t')$ where we alternate distance steps at the first level and global steps, that aggregate a sequence of steps at deeper levels. Then, for all $m \le n$, $j \in \{1,\dots,k\}$ and $r \in \{1,2\}$, we have $t^{m,j,r} = \pi_m(t^{n,j,r})$.*

Proof. Immediate, by definition of projections and telescopic collections. □

Corollary 2. *For all $n \in \mathbb{N}$, $j \in \{1,\dots,k\}$ and $r \in \{1,2\}$, we can decompose $t^{n,j,r}$ as above into $\sum_{i=1}^{I_{njr}} a_i t_i^{n,j,r}$, where the sequences $t^{n,j,1} \rightsquigarrow_{\alpha,d_{nj2}} t^{n,j,2}$ satisfy $I_{nj2} = I_{nj1}\ \forall j \in \{1\dots k\}$, and can be factorized into: $t^{n,j,1} = \sum a_i t_i^{n,j,1} \rightsquigarrow_{\alpha,d_{nj2,1}} \sum a_i t_i^{nj2,1} \cdots \rightsquigarrow_{\alpha,d_{nj2,I_{nj2}}} \sum a_i t_i^{nj2,I_{nj2}} = \sum a_i t_i^{n,j,2} = t^{n,j,2}$, where $t_i^{nj2,l} = t_i^{n,j,2}$ $\forall l \le i$ and $t_i^{nj2,l} = t_i^{n,j,1}$ $\forall l > i$, and $\sum_{i=1}^{I_{nj2}} d_{nj2,i} = d_{nj1}$.*

As a consequence, if we denote by $t^{\infty,j,r}$ the unique tree in $FyTree$ that satisfies $\pi_n(t^{\infty,j,r}) = t^{n,j,r}$ $\forall n \in \mathbb{N}$, we can decompose it into $\sum_{i=1}^{I_{j,r}} a_i t_i^{\infty,j,r}$, and each collection $(\mathscr{S}^n_{j,i})_{n\in\mathbb{N}}$ given by $\mathscr{S}^n_{j,i} := \pi_n(t_i^{\infty,j,1}) = t_i^{n,j,1} \rightsquigarrow_{\alpha,d_{nj2}} t_i^{n,j,2} = \pi_n(t_i^{\infty,j,2})$ is indeed a telescopic collection relating $t_i^{\infty,j,1}$ and $t_i^{\infty,j,2}$.

Proof. Easy to check, by definition of projections and telescopic collections. □

Theorem 1. *Let $\mathscr{D} = \{(t,t',d) \mid \exists\, (\mathscr{S}^n)_{n\in\mathbb{N}}$ a telescopic collection relating t and $t'\}$, then $\mathscr{D}$ is an α-ccd.*

Proof. Using the notation in Lem. 2 and Cor. 2 above, we can consider the coinductive sequence

$$t = t^{\infty,0,1} \equiv_{d_{02}}^{\mathscr{D},\alpha} t^{\infty,0,2} \equiv_{d_{11}}^{\mathscr{D},\alpha} t^{\infty,1,1} \equiv_{d_{12}}^{\mathscr{D},\alpha} \cdots \equiv_{d_{k2}}^{\mathscr{D},\alpha} t^{\infty,k,2} = t',$$

where $d_{j1} = sup_{k\in\mathbb{N}}\{d_{nj1,k}\}$ and $d_{j2} = d_{nj2,k}$, $\forall k \in \mathbb{N}$. It is clear that all the steps $t^{\infty,j,1} \equiv_{d_{j2}}^{\mathscr{D},\alpha} t^{\infty,j,2}$ are valid first level steps, while using Cor. 2 we have that the steps $t^{\infty,j,2} \equiv_{d_{(j+1)1}}^{\mathscr{D},\alpha} t^{\infty,j+1,1}$ correspond to valid coinductive steps from $\mathscr{D}$. This is so because joining all the members of the telescopic sequences $(\mathscr{S}^n_{j,i})_{n\in\mathbb{N}}$ with $j \in \{0\dots k\}$ fixed, we obtain a single telescopic sequence $(\mathscr{S}^n_j)_{n\in\mathbb{N}}$ relating $t^{\infty,j,2} = \sum_{i=1}^{I_{j,r}} a_i t_i^{\infty,j,2}$ and $t^{\infty,(j+1),1} = \sum_{i=1}^{I_{j,r}} a_i t_i^{\infty,j+1,1}$. □

[2]Certainly, these "telescopic" sequences correspond to the notion of inverse limit in domain theory or category theory. But since we only need a very concrete case of that quite abstract notion, we prefer to define it in an explicit way here.

If for any $t, t' \in FyTrees$ there would be only finitely many sequences proving each valid triple $t \rightsquigarrow_{\alpha,d} t'$, then a classical compactness technique (or König's lemma, if you prefer) would immediately prove that whenever we have $\pi_n(t) \equiv_d^{\alpha} \pi_n(t')$ for any $n \in \mathbb{N}$, we can obtain a telescopic collection of sequences proving all these facts. Then, the application of Th. 1 would conclude the continuity of our global bisimulation distance. Unfortunately, this is not the case, because we can arbitrary enlarge any such sequence, adding dummy steps that apply the idempotency rule in one and the other directions. Even more, in some complicated cases in order to obtain some distances between two trees we need to consider sequences that include intermediate trees wider than the compared ones.

Example 4. *Let $\mathbb{A} = \{1,2,3,4,5\}$ with the "usual" distance $\mathbf{d}(n,m) = |m-n|$, for all $m,n \in \mathbb{A}$. Let us consider the trees $t = 1(2+3+4+5)+1(1+2+3+4)$ and $t' = 1(1+2+4+5)$. Then, we have $t \rightsquigarrow_{1,3} t'$, which can be obtained by means of the sequence $\mathscr{S} := t \rightsquigarrow^1_{1,0} 1(2+2+3+4+5)+1(1+2+3+4) \rightsquigarrow^1_{1,1} 1(1+2+3+4+5)+1(1+2+3+4)) \rightsquigarrow^1_{1,0} 1(1+2+3+4+5)+1(1+2+3+4+4) \rightsquigarrow^1_{1,1} 1(1+2+3+4+5)+1(1+2+3+4+5) \rightsquigarrow^1_{1,0} 1(1+2+3+4+5) \rightsquigarrow^1_{1,1} 1(1+2+4+4+5) \rightsquigarrow^1_{1,0} 1(1+2+4+5) = t'$.*

Proposition 4. *There exist $t, t' \in FTrees(\mathbb{A})$, with $t \rightsquigarrow_{\alpha,d} t'$ and $||t||_m, ||t'||_m \leq k$ for some $m, k \in \mathbb{N}$, for which it is not possible to obtain an operational sequence $\mathscr{S}$ witnessing $t \rightsquigarrow_{\alpha,d} t'$ whose intermediate trees t^i satisfy $||t^i||_m \leq k$.*

Proof. For the trees t, t' in Ex. 4, there is no sequence $\mathscr{S}$ proving $t \rightsquigarrow^1_{1,3} t'$ that only includes intermediate trees t'' with $||t''||_2 \leq 4$.

We have to transform the sets $A = \{2,3,4,5\}$ and $B = \{1,2,3,4\}$ into $C = \{1,2,4,5\}$. First of all, we need to unify A and B, otherwise if we try to get C from A and B independently it will cost more than 3 units. In order to unify A and B, we need at least a 2 units payment getting one of the following intermediate sets: $C'_1 = \{1,2,3,4,5\}$, $C'_2 = \{1,2,3,4\}$, $C'_3 = \{2,3,4,5\}$ or $C'_4 = \{2,3,4\}$. Now, we see that C'_1 is the only intermediate set that gives us the 3 units cost given in Ex. 4.

$C'_2 \rightsquigarrow_{1,2} C$: Pay 1 unit to add 5, and another to "erase" 3, so that the total cost would be 4.

$C'_3 \rightsquigarrow_{1,2} C$: Pay 1 unit to add 1, and another to "erase" 3, so that the total cost would be 4 again.

$C'_4 \rightsquigarrow_{1,3} C$: Pay 1 unit to add 1, another to add 5, and another to "erase" 3; the total cost would be 5. □

Also, the following example shows us that in some cases the sequences at the telescopic sequences could be a bit involved. Even if for small values of m the projections $\pi_m(t)$ and $\pi_m(t')$ could be "quite close", we need more elaborated sequences for relating them. Otherwise we would not be able to expand those sequences into others connecting $\pi_n(t)$ and $\pi_n(t')$, for larger values of n.

Example 5. *We can check $ac+bd \rightsquigarrow_{1,2} ad+bc$ by using the sequence $\mathscr{S} := ac+bd \rightsquigarrow^1_{1,1} bc+bd \rightsquigarrow^1_{1,1} bc+ad = ad+bc$, which corresponds to $\pi_1(\mathscr{S}) = \mathscr{S}^1 := a+b \rightsquigarrow^1_{1,1} b+b \rightsquigarrow^1_{1,1} b+a = a+b$. Therefore, in this case, we cannot start with the trivial sequence $\mathscr{S}'^1 := a+b = b+a$, and moreover we need to take the order of "summands" into account, expressing the fact that we have really to change a into b, and b into a.*

5 Some partial results looking for continuity

Ex. 4 shows us that sometimes we need to use intermediate trees containing subtrees that are wider than the corresponding ones in the two compared trees. However, we expected to prove that the width of those subtrees would be bounded by some (simple) function of the width of the subtrees at the same depth of the compared trees. We tried a proof by induction on the depth of the trees, whose base case should correspond to depth 1.

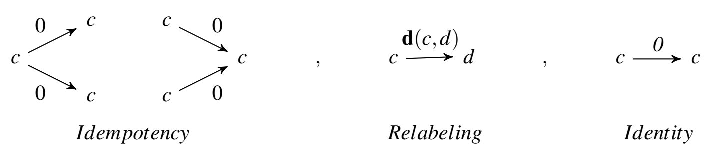


Figure 2: Arcs of the graph representing the transformation of single level trees.

We consider 1-depth trees $t = \sum_{i=1}^{n} a_i$ and $t' = \sum_{j=1}^{m} b_j$, and the corresponding multisets of labels $A = \{a_1,\dots,a_n\}$ and $B = \{b_1,\dots,b_m\}$. Now any step in a sequence $t \rightsquigarrow_{\alpha,d} t'$ is, of course, a first level step and corresponds to the application of either idempotency, in any direction, or the relabeling rule. We can represent these sequences by means of a multi-stage graph. We formally denote these graphs by (S,T,l,v), where (i) S is partitioned into $\{S_j\}_{j=0,\dots,m}$, (ii) $l\mid_{S_0}: S_0 \to A$ and $l\mid_{S_m}: S_m \to B$ are bijections, and (iii) arcs of T are of the form (t,u), with $t \in S_i$ and $u \in S_{i+1}$ for some i, and correspond to any of the patterns in Fig. 2, with the application of a single non-identity pattern at each stage. Then, the full cost d of the sequence is just the sum of the costs of all the arcs in the graph.

Definition 9. *We say that a multi-stage graph is*

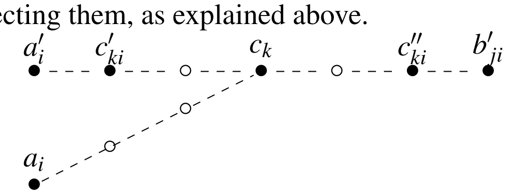
1. Totally sides connecting *(tsc) if for all $s_0 \in S_0$ there exists a path connecting it with some $s_m \in S_m$, and, symmetrically, for all $s_m \in S_m$ there exists a path connecting it with some $s_0 \in S_0$.*
2. Totally both ways connected *(tbwc) if it is tsc and for each $j \in \{1,\dots,m-1\}$ and each $s \in S_j$ there exist two arcs $(s',s),(s,s'') \in T$.*

Proposition 5. *Let $t = \sum_{i=1}^{n} a_i$ and $t' = \sum_{j=1}^{m} b_j$ as above. Then, whenever we have $t \rightsquigarrow_{\alpha,d} t'$, we can prove this by means of a sequence that has at most $3(m+n-2)+1$ distance steps, and thus the width of the trees along it are also bounded by that amount.*

Proof. We consider the multi-stage graph $\mathscr{G}$ that represents the sequence $t \rightsquigarrow_{\alpha,d} t'$. We observe that this graph is tsc, using this fact we obtain a "compacted" subgraph $\mathscr{H}$ (of the original graph $\mathscr{G}$) by selecting a subset of the nodes of $\mathscr{G}$ and disjoints paths connecting them. So that, ever node in $\mathscr{G}$-$\mathscr{H}$ is at most in one of theses paths.

We turn these paths into arcs of $\mathscr{H}$ whose cost is $d(c_i,c_j)$, where c_i, c_j are the two extremes of the path. By applying the triangle inequality, we immediately obtain that this cost is no larger than the cost of the original path. Therefore, the full cost of $\mathscr{H}$ is no bigger than that of $\mathscr{G}$. The set of nodes in $\mathscr{H}$ is obtained in the following way:

- First, we consider $a_1 \in A$ and some reachable $b_{j1} \in B$ from it. We introduce both of them in $\mathscr{H}$, and also add the arc connecting them, as explained above.



- Next, we consider each of the remaining $a_i \in A$ and we select again some b_{ji} with which it is connected. We take the path in $\mathscr{G}$ connecting them, and if it does not cross any path in $\mathscr{G}$ that generated an arc in $\mathscr{H}$, then we proceed as in the first case. Otherwise, we consider the first arc $\langle c'_{ki}, c''_{ki}\rangle$ in $\mathscr{H}$ "traversed" by the new path. If c_{ki} is the common node to the two involved paths, then we add it to $\mathscr{H}$ and remove the arc $\langle c'_{ki}, c''_{ki}\rangle$ adding instead two new arcs $\langle c'_{ki}, c_{ki}\rangle$ and $\langle c_{ki}, c''_{ki}\rangle$, together with the arc $\langle a_i, c_{ki}\rangle$.

Finally, we proceed in the same way, but going backwards in the graph, for every $b_j \in B$ that was not still reached from any $a_i \in A$. Clearly, at any stage of the construction we add two new arcs in the worst case, and besides we need an idempotency step each time we consider a path that crosses $\mathscr{H}$. This finally produces a sequence from t to t' with at most $3(m+n-2)+1$ steps, whose cost is not bigger than that of the original sequence. Since the width of the trees change at most one at any step, the results about the bound of these widths follows immediately. $\square$

The important thing about the bounds obtained above, is that they only depend on the cardinality of the multisets A and B, but no at all on the properties of the domain of actions $(\mathbb{A}, \mathbf{d})$. Even more, we can extend this result to any two finite trees $t = \sum_{a \in A} a t_a$ and $t' = \sum_{b \in B} b t_b$: the size and complexity of the "continuations" t_a and t_b do not compromise the bound on the number of first level steps of a sequence relating t and t', that bound only depends on $||t||_1$ and $||t'||_1$.

Proposition 6. *Let $t = \sum_{i=1}^{n} a_i t_i$ and $t' = \sum_{j=1}^{m} b_j t_j$ be two trees such that $t \equiv_d^\alpha t'$. Then, we can prove this by means of a coinductive transformation sequence $\mathscr{C}$, that has at most $3(m+n-2)+1$ first level steps.*

Proof. We observe that the result in Prop. 5 could be obtained in exactly the same way if instead of a distance function on A, we would consider a relation $d \subseteq A \times A \times \mathbb{R}^+$ that satisfies the properties that define "bounds for a distance" in A:

- $\forall a \in A \; d(a,a,d) \; \forall d \in \mathbb{R}^+$.
- $d(a,b,d) \Leftrightarrow d(b,a,d)$.
- $d(a,b,d_1) \wedge d(b,c,d_2) \Rightarrow d(a,c,d_1+d_2)$.

Then, we consider the set of "prefixed" trees $\mathbb{A}FTrees = \mathbb{A} \times FTrees$, and we take $d_\alpha(at, bt', d_1+d_2)$ $\Leftrightarrow d_1 = \mathbf{d}(a,b) \wedge t \rightsquigarrow_{\alpha, d_2/\alpha} t'$. It is clear that for any distance d on A, and any $\alpha \in \mathbb{R}^{\cdot}$, each such d_α defines bounds for a distance in A. Now, we can see each finite trees $t = \sum_{i=1}^{n} a_i t_i$ as a 1-depth trees $t = \sum_{i=1}^{n} \langle a_i, t_i \rangle$, for the alphabet $\mathbb{A}FTrees$. For any $\alpha \in \mathbb{R}^+$ we can consider the "bound for a distance" relation d_α, and apply Prop. 5 to conclude the proof. $\square$

5.1 An alternative proof of the bounds for the first level

Even if the bounds obtained in Prop. 5 are rather satisfactory we have been able to prove some tighter bounds by reducing the induced graph by means of local simplifications rules. We consider interesting to show this proof because they bring to light how we could proceed when transforming the trees all along the sequences (even if we have not been able to successfully use this ideas in order to fully prove the desired continuity result).

Proposition 7. *Any multi-stage graph that is tsc can be turned into a tbwc multi-stage graph that is a subgraph of the original one.*

Proof. We repeatedly remove any intermediate node which is not two ways connected, and it is clear that after any such removal the graph remains tsc. $\square$

Proposition 8. *1. The multi-stage graph associated to a sequence proving that $A \equiv_d^\alpha B$ is tbwc.*

2. Any subgraph of such a multi-stage graph, obtained by the removal of some internal nodes, that still remains tbwc, can also be obtained from some sequence proving $A \equiv_d^\alpha B$, which will be shorter or have the same length than the initial sequence.

Proof. (1) Trivial. (2) When removing an internal node we are possibly removing an application of idempotency that turns into a simple identity arc. In such a case, we can remove this stage of the sequence getting a shorter sequence still proving $A \equiv_d^\alpha B$. $\square$

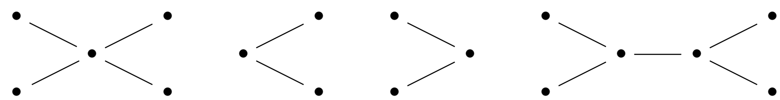


Figure 3: Schematic examples of diabolos

Remark 2. *The result in Prop. 8.2 is true, not only if we remove nodes that are not both ways connected (in fact, there is no such in a tbwc multi-stage graph!), but also if we remove any subset of intermediate nodes, as far as the graph remains tsc. This is what we next use in order to reduce the size of the graph.*

Definition 10. *We say that a tbwc multi-stage graph is a* diabolo *if there exists some stage $i \in \{0 \dots m\}$ with $|S_i| = 1$, such that: (i) for all $j < i$ and all $s_j \in S_j$, $|\{s' \mid (s_j, s') \in T\}| = 1$, and (ii) for all $j > i$ and all $s_j \in S_j$, $|\{s' \mid (s', s_j) \in T\}| = 1$. We say that $s_i \in S_i$ is a center of the diabolo.*

Next we prove that we can reduce any tbwc multi-stage graph into a disjoint union of diabolos. We will do it by removing some "redundant" arcs, in such a way that these removals do not destroy the *tsc* character of the graph.

Definition 11. 1. *We say that a sequence of consecutive arcs*

$$(s_j, s_{j+1}), (s_{j+1}, s_{j+2}) \dots (s_{j+k-1}, s_{j+k})$$

in a tsc graph is a join sequence *if: (i) for all intermediate nodes in the sequence s_{j+l} with $l \in \{1, \dots, k-1\}$, the two arcs in the sequence, (s_{j+l-1}, s_{j+l}) and (s_{j+l}, s_{j+l+1}) are the only ones in T that involve s_{j+l}; and (ii) there are at least two other arcs (s_j, s') and (s'', s_{j+l}) in T that are not in the sequence.*

2. *We say that a node $s_j \in S_j$ is left-reducible (resp. right-reducible) if it can only be reached from a single node $s_0 \in S_0$ (resp. $s_m \in S_m$), but there are several arcs $(s'_{j-1}, s_j) \in T$ (resp. $(s_j, s'_{j+1}) \in T$).*

Join Sequence *Left-reducible Node*

Proposition 9. 1. *Whenever we have a join sequence in a tbwc graph, we can remove all its arcs and the intermediate nodes, preserving the tbwc property.*

2. *Whenever we have a left (resp. right)-reducible node s_j in a tbwc, we can remove one of the arcs (s'_{j-1}, s_j) reaching (resp. (s_j, s'_{j+1}) leaving) the node, preserving the tsc property.*

Proof. (1) Clearly, after the removal the graph remains tbwc, due to the existence of the two "lateral" arcs (s_j, s'_{j+1}) and (s'_{j+l-1}, s_{j+l}). (2) It is clear that after the removal we have still another arc reaching (resp. leaving) the reducible node from the same side. This can be still used to reach the corresponding node $s_0 \in S_0$ (resp. $s_m \in S_m$). And no other node in either S_0 and S_m is affected by the removal. □

Theorem 2. *By removing some intermediate nodes we can reduce any tbwc multi-stage graph into another such graph (with smaller or equal total cost) which is the disjoint union of a collection of diabolos.*

Proof. We start by reducing the graph applying Prop. 9, until we cannot apply it anymore. We obtain a tsc graph connecting the same sets of nodes S_0 and S_m, which can be turned into another tbwc (smaller) by applying Prop.7. By abuse of notation, let us still denote by $S_1,\dots,S_{m-1}$ the other stages of the graph.

- First, we consider the connected components of the graph containing exactly one node s_0 belonging to S_0. These components are right-degenerated diabolos with s_0 as center. Indeed, if this was not the case, then the component would still contain a left-reducible node.
- Now, let us consider a connected component containing a subset of nodes $\{s_0^1,\dots,s_0^k\} \subseteq S_0$, with $k>1$. It is not difficult to verify that for each s_0^i $(1 \leq i \leq k)$, there is exactly one arc leaving s_0^i: otherwise, the connected component would still contain a join sequence (due to the presence of multiple nodes in S_0). Using a similar argument, it is possible to prove that there can be only a single arc leaving each node in the following stages as well, until a stage i is reached such that the component contains exactly one node $s_i \in S_i$. Reasoning in a symmetric way from the right side (set S_m), it is eventually possible to show that the considered component is a diabolo.

Therefore, after the reduction, the graph is indeed the union of a family of disjoint diabolos. □

Proposition 10. *1. A diabolo connecting two sets of nodes S_0 and S_m satisfies $|S_i| \leq max\{|S_0|,|S_m|\}$ for each i.*

2. Any disjoint union of diabolos connecting S_0 and S_m satisfies $|S_i| \leq |S_0|+|S_m|$ for each i.

3. If the multi-stage graph corresponding to a sequence proving $A \equiv_d^\alpha B$ is a diabolo, then we can obtain another such sequence whose length will be at most $3(|A|+|B|-2)+1$.

Proof. (1) Obvious. (2) We could think that this is an immediate consequence of 1, but this is not always the case. Let us consider the disjoint union S of two three-stages (degenerated) diabolos S^1 and S^2, with $|S_0^1| = 1 = |S_2^2|$; $|S_0^2| = 8 = |S_2^1|$; $|S_1^1| = 5 = |S_1^2|$. Then we have $S_0 = S_0^1 \cup S_0^2$, $S_1 = S_1^1 \cup S_1^2$, $S_2 = S_2^1 \cup S_2^2$ and therefore $|S_1| = 10$, but $|S_0| = |S_2| = 9$. As a consequence, the result would be wrong if we put *max* instead of $+$. However, what we asserted is true in general: Let S the disjoint union of a family of diabolos S^j with $j \in \{1..k\}$, then we have $S_i = \bigcup_{j=1}^k S_i^j$, and as a consequence of 1, $|S_i^j| \leq |S_0^j| + |S_m^j|$, and therefore $|S_i| = \sum_{j=1}^k |S_i^j| \leq \sum_{j=1}^k |S_0^j| + \sum_{j=1}^k |S_m^j| = |S_0| + |S_m|$. (3) Whenever we have two relabeling steps at the same place, with no idempotency steps between them affecting that place, we can join them into a single relabeling step, without increasing the cost of the full transformation, because of triangular transitivity. As a consequence, we would have at most two relabeling steps for each idempotency step, and we have exactly $|A|+|B|-2$ idempotency steps at each diabolo. So we have, at the moment, at most $3(|A|+|B|-2)$ steps; possibly, we will need only one more relabeling step at the center of the diabolo. This gives us the bound $3(|A|+|B|-2)+1$. □

Corollary 3. *If we can prove $A \equiv_d^\alpha B$, then we can do it by means of a sequence whose intermediate sets satisfy $|S_i| \leq |A|+|B|$, and has at most $3(|A|+|B|-2)+1$ steps.*

Proof. We apply Th. 2 to reduce the multi-stage graph corresponding to the sequence proving $A \equiv_d^\alpha B$. Then apply Prop. 10 to get the bounds for the values $|S_i|$, and that for the number of steps. □

5.2 Now we go down into the second level

Once we have the base case of an inductive proof, we would like to proceed with the inductive case. When we thought that we had it, we decided to present first the particular case of the trees with only two level, because its (bigger) simplicity would help the readers to understand the quite involved proof for the general case. Next we present the proof for this particular case. Starting from $p = \sum_{i \in I} a_i p_i$ and

$q = \sum_{j\in J} b_j q_j$, once we have proved our Cor. 3, we can assume that the first level steps in the sequences proving $\pi_n(p) \equiv^{\alpha}_{d} \pi_n(q)$ are always the same and satisfy the bounds in the statement of Th. 2. In order to study in detail the second level steps in these sequences, we need to study how the summands p_i evolve along those sequences.

Once we have some bound for the width of any process p' obtained by the evolution of the summands p_i, and another one for the length of the subsequences producing their evolution, adding all these bounds and that corresponding to the first level, we obtain the bound for the two first levels together. The following Prop. 11 proves a preliminary result. It says that whenever we have a sequence proving $p \equiv^{\alpha}_{d} q$ with a "limited number" of first level steps, but q contains summands aq_i where $||q_i||$ is "very large", then we can prune these summands, getting some q' for which $p \equiv^{\alpha}_{d} q'$ can be proved by means of a sequence that only contains processes p^i for which $||p^i||_2$ is "moderately large", and for any process q'' "between" q' and q (that means that q'' can be obtained by adding some branches to some subprocesses of q', and also q can be obtained from q'' in this way) we also have $p \equiv^{\alpha}_{d} q''$. In order to make easier its comprehension, we first present a lemma that covers a particular case corresponding to intermediate stages of a sequence connecting two processes p and q with $||p||$ and $||q||$ "small".

Lemma 3. *For any intermediate process p^i in the sequence $\mathscr{C}$ proving $p = \sum_{i=1}^{n} a_i p_i \equiv^{\alpha}_{d} q = \sum_{j=1}^{m} b_j q_j$, we can decompose p^i into $p^i = p^{i1} + p^{i2}$ in such a way that $||p^{i1}|| \leq ||p|| + ||q||$, and we have a sequence $\mathscr{C}' := p \equiv^{\alpha}_{d_1} p^{i1} \equiv^{\alpha}_{d_2} q$ with $d = d_1 + d_2$, which has at most $3(|I| + |J| - 2) + 1$ first level steps, and only uses intermediate processes $r = \sum_{k\in K} c_k r_k$ with $||r|| \leq ||p|| + ||q||$.*
Moreover, for any decomposition $p^{i2} = p^{i3} + p^{i4}$, we can obtain a sequence $\mathscr{C}'' := p \equiv^{\alpha}_{d_1} p^{i1} + p^{i3} \equiv^{\alpha}_{d_2} q$ which has at most $(3(|I| + |J| - 2) + 1) * (||p^{i3}|| + 1)$ first level steps, and only uses intermediate processes $r = \sum_{k\in K} c_k r_k$, with $||r|| \leq ||p|| + ||q|| + ||p^{i3}||$.

Proof. The first part is in fact a new formulation of Th. 2, observing that each node at the i-th stage of the multi-stage graph induced by $\mathscr{C}$ corresponds to a summand of the corresponding process p^i. When reducing the multi-stage graph, we are pruning some of these summands. Therefore, the obtained intermediate processes at the sequence $\mathscr{C}'$ are the processes p^{i1} of the searched decomposition of p^i. Then, the remaining summands in p^{i2} correspond to the nodes from the i-th stage of the multi-stage graph that were removed when reducing it. It is easy to see that any of these summands can be "reset" into the multi-stage graph by means of a path that will connect the corresponding node with the two sides of the original multi-stage graph. This requires $(3(|I| + |J| - 2) + 1)$ additional arcs, and therefore $(3(|I| + |J| - 2) + 1)$ more steps in the sequence $\mathscr{C}''$, while the width of the intermediate processes in it is increased at most by 1, when adding each of those paths. □

Proposition 11. *For each $f \in \mathbb{N}$, there is a constant $C_{2,f}$ such that if we have a sequence $\mathscr{C}$ proving $p \equiv^{\alpha}_{d} r$ with f first level steps and $r = \sum_{k\in K} c_k r_k$, with $r_k = \sum_{l\in K_k} c_{k,l} r_{k,l}$, then we can obtain some $r' = \sum_{k\in K} c_k r'_k$ with $r'_k = \sum_{l\in L_k} c_{k,l} r'_{k,l}$, where $L_k \subseteq K_k$, with $|L_k| \leq 2^f ||p||$, whose intermediate processes p^i are of the form $p^i = \sum a^i_j p^i_j$, with $p^i_j = \sum_{k\in K^i_j} b_k p^i_{j,k}$ and $|K^i_j| \leq 2^f ||p||$, and $length(S') \leq C_{2,f}$.*
Moreover, taking $m \in \mathbb{N}$, we have also a family of constants $C_{2,f,m}$ such that for any $r'' = \sum c_k r''_k$ with $r''_k = \sum_{l\in L'_k} c_{k,l} r_{k,l}$, where $L_k \subseteq L'_k \subseteq K_k$, we can also prove $p \equiv^{\alpha}_{d} r''$, by means of a sequence $\mathscr{C}''$ whose intermediate processes p^i satisfy $||p^i||_2 \leq max\{2^f ||p||, |L'_k|_{k\in K}\}$, and $length(S'') \leq C_{2,f,max\{|L'_k|_{k\in K}\}}$.

Theorem 3. *For all $k \in \{1,2\}$ and $w \in \mathbb{N}$ there exists a bound $lb(k,w) \in \mathbb{N}$ such that for all p, q with $||p||_k$, $||q||_k \leq w$ and $p \equiv^{\alpha}_{d} q$ we can prove this by means of a sequence $\mathscr{C}'$ that has no more than $lb(k,w)$ steps in each of their two first levels.*

Proof. At the same time that the existence of the bound $lb(k,w)$ we will prove a bound $wb(k,w)$ for the width $||p^i||_2$ of any process p^i along the 2-unfolding of the sequence $\mathscr{C}'$.
<u>k=1</u> Th. 2 just states the result for this case, taking $wb(1,w) = 2w$ and $lb(1,w) \leq 6w$.
<u>k=2</u> Let $\mathscr{C}$ be a sequence proving $p \equiv_d^\alpha q$ which has less than $lb(1,w)$ steps at the first level. If $\mathscr{C}$ contains no first level step, then we have $p = \sum_{i=1}^n a_i p_i$, $q = \sum_{i=1}^n a_i q_i$ and $\mathscr{C}$ can be factorized into a collection of sequences $(\mathscr{C}_i')_{i=1}^n$ proving $p_i \equiv_{d_i} q_i$. Then, we only need to apply Th. 2 to each of these sequences, so that we could take $wb(2,w) = 2w$ and $lb(2,w) = 2w * 6w = 12w^2$.
If $\mathscr{C}$ contains some first level step, we select any of them and partition the 2-unfolding of $\mathscr{C}$ into $\mathscr{C}^1 \circ \langle s \rangle \circ \mathscr{C}^2$, where by abuse of notation we identify the sequences and their 1-unfolding. Let us consider the case in which the central step corresponds to a relabeling $ap_i' \to bp_i'$ (the other cases are analogous). Assume that $\mathscr{C}^1$ proves $p \equiv_{d_1}^\alpha p'$ and $\mathscr{C}^2$ proves $q' \equiv_{d_3}^\alpha q$, with $d = d_1 + \mathbf{d}(a,b) + d_3$. We can apply Prop. 11 to both sequences, taking $\mathscr{C}^2$ in the opposite direction, getting r' and r'', with $\mathscr{C}'^1$ proving $p \equiv_{d_1}^\alpha r'$ and $\mathscr{C}'^2$ proving $r'' \equiv_{d_3}^\alpha q$, that have the same steps at the first level than $\mathscr{C}^1$ and $\mathscr{C}^2$, respectively, and only use intermediate processes p^i with $||p^i||_2 \leq 2^{lb(1,w)-1} * w$. The result is also valid for any intermediate process between r_1 and p', and anyone between r_2 and q', but increasing the bounds in the adequate way. In particular, for the process $r_1 + r_2$, we have $p \equiv_{d_1}^\alpha r_1 + r_2$ and $r_1 + r_2 \equiv_{d_3}^\alpha q$, by means of sequences that only include intermediate processes p^i with $||p^i||_2 \leq 2^{lb(1,w)} * w$. Joining these two sequences with the step s we obtain the sequence $\mathscr{C}'$. Adding the bounds for the corresponding lengths of the 2-unfolding of the two sequences obtained by application of Prop. 11, we obtain the bound $lb(2,w)$. To be more precise, the definitive value of $lb(2,w)$ will be the maximum of the bounds for the different cases (there are finitely many) considered above. □

Certainly, the proof is quite involved and difficult to follow, but it works in this case. Unfortunately, when we tried to present it for the general case we discovered that some details failed, or at least cannot be justified in the same way. Let us roughly explain why: the argument used in Lemma 3 looks for an small "kernel" p^{i1} of any intermediate process along a sequence. Then Prop. 11 joins the kernels obtained when considering the two subsequences starting at any side of the original sequence and ending at any intermediate process. The second part of Lemma 3 says us that we can get in this way a sequence satisfying the desired bounds. But when we are in a deeper level, then a duplication (by using idempotency) at a lower level could cause an undesired increasing of the "closure" of the union of the two (one from each side) kernels. And this problem could appear over and over (at least, we are not able to prove that this is not the case). Therefore, we have to leave this problem open, at the moment, any help?

6 Conclusions and future work

The two first authors of this paper are involved in a detailed study of bisimulation distances, that will constitute in fact the forthcoming Ph.D. Thesis of the first one, supervised by the second. As we have already stated at the paper, it is true that our global bisimulation distance is more difficult to manage than the classical bisimulation distance, but instead we think that it produces a much more accurate measuring of the differences between two processes. Moreover, our approach has many nice properties, and we still think that continuity with respect to projections will be one of them. However, at the moment we have not been able to present a full proof, but once we have invested a great number of hours in it, we think that it is time to present all our (partial) results on the subject.

We are now working on several extension of our approach. In particular the modal interface framework [10, 9, 13] is a quite suggestive one, where we are obtaining several quite promising results. Also, the stochastic distances [8] and those based on logics [5] are interesting topics. In fact, when considering probabilistic choices between branches starting with the same action, its is true that those proposals somehow apply our global approach, but instead this is not at all the case when choices between different actions are considered. We would like to provide a "fully global" stochastic distance that will satisfactory extend our global distance to this framework.

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