Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018).
https://hdl.handle.net/11705/PROLE/2018/009

Directions of Operational Termination

Salvador Lucas

DSIC, Universitat Politècnica de València, Spain

A theory $\mathscr{S}$ in a logic supplied with an inference system is operationally terminating if no goal has an infinite *well-formed* proof tree. Well-formed proof trees are those which an interpreter would incrementally build when trying to solve a condition at a time from left to right. For this reason, *infinite* well-formed proof trees have a *unique* infinite branch which is called the *spine*. This paper introduces the notion of a *directed* proof tree for $\mathscr{S}$ and a set of formulas Δ, which we call a *direction*. Intuitively, a direction Δ is intended to collect formulas that are infinitely often used in the spine of an infinite well-formed proof tree (which is then called Δ-*directed*) due to the repeated use of some specific inference rules. Then we introduce the notion of Δ-*directed operational termination* of a theory as the absence of Δ-directed proof trees. This new notion permits the definition of different termination properties which can be useful to distinguish different computational behaviors. It also gives a new characterization of operational termination of a (finite) theory $\mathscr{S}$ as the conjunction of the Δ-directed operational termination of $\mathscr{S}$ for each direction Δ in a (finite) set of directions.

Keywords: General Logics, Operational Termination, Program Termination

1 Introduction

In the semantic description of computational systems, the use of *inference systems* is pervasive. Inference systems (i) provide a basis for the *description* of computational systems and computational relations, and (ii) can also be used for the (often prototypic) *implementation* of such computational systems. With regard to the descriptive aspect, we can *define* computational relations by *provability* of specific goals. The *proof* of a goal with respect to the inference system is often witnessed by a *proof tree*.

Example 1 *The one-step relation for a Term Rewriting System (TRS, [2]) $\mathscr{R}$ over a signature $\mathscr{F}$ of symbols f with an arity $ar(f)$ and rewrite rules $\alpha : \ell \to r$ with label α and terms ℓ, r can be defined by the inference system $\mathscr{I}_1(\mathscr{R})$ with the following inference rules for each rule α, function symbol $f \in \mathscr{F}$ and $1 \leq i \leq k = ar(f)$:*

$$(Rl)_\alpha \ \frac{}{\ell \to r} \qquad (C)_{f,i} \ \frac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to f(x_1,\ldots,y_i,\ldots,x_k)}$$

Remark 1 *In this paper, inference rules $\frac{B_1 \ \cdots \ B_n}{A}$ can be used for any* instance *$\frac{\sigma(B_1) \ \cdots \ \sigma(B_n)}{\sigma(A)}$ of the rule by a substitution σ [24]. For instance, $(Rl)_\alpha$ establishes that every instance $\sigma(\ell)$ of ℓ rewrites into $\sigma(r)$.*

The one-step rewrite relation *$\to_{\mathscr{R}}$ for $\mathscr{R}$ is defined by provability in $\mathscr{I}_1(\mathscr{R})$:*

$$\to_{\mathscr{R}} \ = \ \{(s,t) \mid s \text{ and } t \text{ are terms, and } \mathscr{I}_1(\mathscr{R}) \vdash s \to t\}$$

where $\mathscr{I}_1(\mathscr{R}) \vdash s \to t$ means that the goal $s \to t$, where s and t are terms, can be proved in $\mathscr{I}_1(\mathscr{R})$. Similarly, we define the transitive closure $\to^+_{\mathscr{R}}$ of $\to_{\mathscr{R}}$ by provability in a new inference system $\mathscr{I}_2(\mathscr{R})$ which is obtained by just adding *the following two rules to $\mathscr{I}_1(\mathscr{R})$:*

$$(Sq_1^+) \ \frac{x \to y}{x \to^+ y} \qquad (Sq_2^+) \ \frac{x \to y \qquad y \to^+ z}{x \to^+ z}$$

Submitted to:
PROLE 2018

© Salvador Lucas
This work is licensed under the
Creative Commons Attribution License.

Still, we can give an equivalent definition of $\to^+_{\mathscr{R}}$ *by using the following* smaller *inference system* $\mathscr{I}_3(\mathscr{R})$*:*

$$(Rl^+)_\alpha\ \frac{}{\ell \to^+ r} \qquad (C^+)_{f,i}\ \frac{x_i \to^+ y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to^+ f(x_1,\ldots,y_i,\ldots,x_k)} \qquad (T^+)\ \frac{x \to^+ y \qquad y \to^+ z}{x \to^+ z}$$

Some knowledge about the properties of computational relations R on terms which are defined by provability of goals $s\ R\ t$ with respect to a theory $\mathscr{S}$ (i.e., by $\mathscr{S} \vdash s\ R\ t$) should be obtained by inspecting the inference systems. For instance, the *well-foundedness* of R, i.e., the absence of infinite sequences $t_1, t_2, \ldots$ of terms such that $t_i\, R\, t_{i+1}$ for all $i \geq 1$. Roughly speaking, R can be proved *well-founded* by just finding a model $\mathscr{A}$ of $\mathscr{S}$ such that the binary relation $R^{\mathscr{A}}$ on $\mathscr{A}$ interpreting predicate R is well-founded [11, Section 5.1]. Tools like AGES [6] can be used to obtain such interpretations. Both $\mathscr{I}_2(\mathscr{R})$ and $\mathscr{I}_3(\mathscr{R})$ in Example 1 could be used to (try to) obtain a proof of well-foundedness of $\to^+_{\mathscr{R}}$ in this way.

On the other hand, whenever an inference system is used to *implement* computations, the question about how such an implementation behaves from an *operational* point of view naturally arises and, again, some information about such an *operational behavior* should be obtained from the inference system. The notion of operational termination as the absence of goals with an infinite *well-formed* proof tree [13] exploits this idea regarding the *termination behavior* of computational systems whose implementation is based on proving goals as explained above. Well-formed proof trees are those which an interpreter would incrementally (and quite naturally) build when trying to solve a condition at a time from left to right. We note that the well-foundedness of binary relations defined by provability and the operational termination of the corresponding inference system may *differ*.

Example 2 *Given a TRS* $\mathscr{R}$*,* $\mathscr{I}_2(\mathscr{R})$ *is* operationally terminating *iff* $\to^+_{\mathscr{R}}$ *is well-founded. However, for all TRSs* $\mathscr{R}$*,* $\mathscr{I}_3(\mathscr{R})$ *is operationally* nonterminating*: the infinite well-formed proof tree*

$$\dfrac{\dfrac{\dfrac{\vdots}{s \to^+ s_2}(T^+) \qquad s_2 \to^+ s_1}{s \to^+ s_1}(T^+) \qquad s_1 \to^+ t}{s \to^+ t}(T^+)$$

(where t*,* s*,* $s_1, \ldots$ *are arbitrary terms) is obtained in* $\mathscr{I}_3(\mathscr{R})$ *by a repeated use of rule* (T^+) *alone. Thus, well-foundedness of* $\to^+_{\mathscr{R}}$ *is* not *equivalent to operational termination of* $\mathscr{I}_3(\mathscr{R})$*, although it defines* $\to^+_{\mathscr{R}}$*.*

Example 2 shows a 'mismatch' between operational termination and well-foundedness as *basic notions* to capture the termination behavior of computational relations. In this paper we introduce a refinement of the notion of operational termination which is helpful to solve this problem. After some preliminaries in Section 2, we introduce the notion of *directed* proof tree. Roughly speaking, given a set of formulas Δ (called a *direction*), an infinite (well-formed) proof tree is said to be Δ-*directed* if only inference rules $\frac{B_1 \cdots B_n}{A}$ whose head A is *subsumed* by a formula in Δ are used in the only possible infinite branch (called the *spine*). We use the term '*direction*' as a hopefully suggestive way to classify all possible *infinite paths* we may follow in infinite proof trees associated to a given (finite) inference system. For instance, the infinite proof tree above is $\{x \to y\}$-directed. The *absence* of Δ-directed proof trees is called Δ-*directed operational termination* (Section 3). We show how to use this new termination property to deal with well-known approaches (e.g., well-foundedness, or dependency pairs [1]) to define termination properties of computational systems and programming languages (Section 4). For instance, we can prove well-foundedness of $\to^+_{\mathscr{R}/E}$ above as directed operational termination of an extension of $\mathscr{I}_{RL}(\mathscr{R}) \cup \mathscr{I}_{EL}(E)$. We can also prove relative termination of a TRS $\mathscr{R}_1$ with respect to another TRS $\mathscr{R}_2$ [3] and termination of TRSs $\mathscr{R}$ using dependency pairs [1] as directed operational termination of appropriate extensions of theories. We also adapt and improve the OT-Framework in [14] to prove directed operational termination (Sections 5 and 6).

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018).
https://hdl.handle.net/11705/PROLE/2018/009

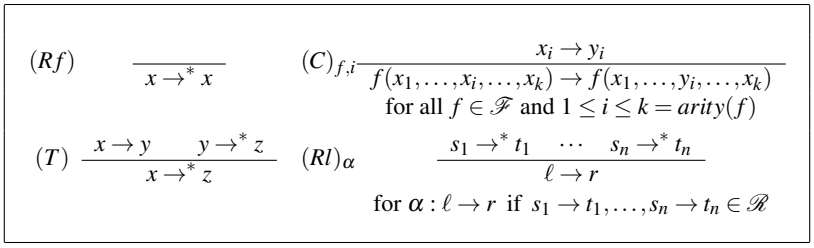


Figure 1: Schemes of inference rules for conditional rewriting

2 Operational Termination in General Logics

A *logic* is a quadruple $\mathscr{L} = (Th(\mathscr{L}), Form, Sub, \mathscr{I})$, where: (i) $Th(\mathscr{L})$ is the class of *theories* of $\mathscr{L}$, (ii) *Form* is a mapping sending each theory $\mathscr{S} \in Th(\mathscr{L})$ to a set $Form(\mathscr{S})$ of *formulas* of $\mathscr{S}$, (iii) *Sub* is a mapping sending each $\mathscr{S} \in Th(\mathscr{L})$ to its set $Sub(\mathscr{S})$ of *substitutions*, with a containment $Sub(\mathscr{S}) \subseteq [Form(\mathscr{S}) \to Form(\mathscr{S})]$, and (iv) $\mathscr{I}$ is a mapping sending each $\mathscr{S} \in Th(\mathscr{L})$ to a subset $\mathscr{I}(\mathscr{S}) \subseteq Form(\mathscr{S}) \times Form(\mathscr{S})^*$, where each $(A, B_1 \ldots B_n) \in \mathscr{I}(\mathscr{S})$ is called an *inference rule* for $\mathscr{S}$ and denoted $\frac{B_1 \ldots B_n}{A}$. Sequences $B_1 \ldots B_n$ of formulas are often written $\vec{B}_n$ for short.

Example 3 *For $\mathscr{L} = CTRS$ (the logic of CTRSs), (i) $Th(CTRS)$ is the class of CTRSs $\mathscr{R} = (\mathscr{F}, R)$ with $\mathscr{F}$ a signature and R a set of conditional rules (over $\mathscr{F}$); (ii) $Form(\mathscr{R}) = \{s \to t, s \to^* t \mid s,t \in \mathscr{T}(\mathscr{F},\mathscr{X})\}$, where $\mathscr{T}(\mathscr{F},\mathscr{X})$ is the $\mathscr{F}$-term algebra with variables $\mathscr{X}$; (iii) $Sub(\mathscr{R}) = \{\lambda\varphi.\sigma(\varphi) \mid \sigma \in Hom_{\mathscr{F}}[\mathscr{T}(\mathscr{F},\mathscr{X}) \to \mathscr{T}(\mathscr{F},\mathscr{X})]\}$, where $Hom_{\mathscr{F}}[\mathscr{T}(\mathscr{F},\mathscr{X}) \to \mathscr{T}(\mathscr{F},\mathscr{X})]$ denotes the set of $\mathscr{F}$-endomorphisms of $\mathscr{T}(\mathscr{F},\mathscr{X})$, and where $\sigma(s \to t) = \sigma(s) \to \sigma(t)$, and $\sigma(s \to^* t) = \sigma(s) \to^* \sigma(t)$; and (iv) $\mathscr{I}(\mathscr{R})$ is the instantiation of the generic inference system of Figure 1.*

We assume some standard properties about the substitutions $\sigma \in Sub(\mathscr{S})$ [14, Section 2]. Given $\phi, \varphi \in Form(\mathscr{S})$, $mgu_{\mathscr{S}}(\phi,\varphi) \subseteq Sub(\mathscr{S})$ denotes a set such that: (i) $\forall \sigma \in mgu_{\mathscr{S}}(\phi,\varphi)$, $\sigma(\varphi) = \sigma(\phi)$; and (ii) $\forall \tau \in Sub(\mathscr{S})$ such that $\tau(\phi) = \tau(\varphi)$, there is $\sigma \in mgu_{\mathscr{S}}(\phi,\varphi), \theta \in Sub(\mathscr{S})$ such that $\tau = \theta \circ \sigma$.

Given a logic $\mathscr{L}$ and a theory $\mathscr{S} \in Th(\mathscr{L})$, a finite proof tree T is either: (i) an *open goal*, simply denoted as G, then, we denote $root(T) = G$; otherwise, (ii) is a *derivation tree* with root G, denoted as $\frac{T_1 \quad \cdots \quad T_n}{G}(\rho)$, where G is as above, $T_1,\ldots,T_n$ are finite proof trees (for $n \ge 0$), and $\rho : \frac{B_1 \cdots B_n}{A}$ is an inference rule such that $G = \sigma(A)$, $root(T_1) = \sigma(B_1), \ldots, root(T_n) = \sigma(B_n)$ for some substitution σ. We write $root(T) = G$. A finite proof tree T is *closed* if it contains no open goals. The *theorems* of $\mathscr{S}$ are the formulas $\varphi \in Form(\mathscr{S})$ for which we can derive a *closed* proof tree.

A finite proof tree T is a *proper prefix* of a finite proof tree T' (written $T \subset T'$) if there are one or more open goals $G_1, \ldots, G_n$ in T such that T' is obtained from T by replacing each G_i by a derivation tree T_i with root G_i. An *infinite proof tree* T is an infinite increasing chain of finite proof trees, i.e., a sequence $(T_i)_{i \in \mathbb{N}}$ such that for all i, $T_i \subset T_{i+1}$. Since for all $i \in \mathbb{N}$, $root(T_i) = root(T_{i+1})$, we write $root(T) = root(T_0)$. There can be different (equivalent) ways to represent an infinite proof tree T [15].

A finite proof tree T is *well-formed* if it is either an open goal, or a closed proof tree, or a derivation tree $\frac{T_1 \quad \cdots \quad T_n}{G}(\rho)$, where $T_1, \ldots, T_{i-1}$ are closed for some $1 \le i \le n$, T_i is a well-formed but not closed finite proof tree, and $T_{i+1}, \ldots, T_n$ are open goals. Note the *left-to-right* construction of the proof tree. An infinite proof tree is *well-formed* if it is an increasing chain of well-formed finite proof trees.

Definition 1 (Operational termination [13]) *A theory $\mathscr{S}$ in a logic $\mathscr{L}$ is called* operationally terminating *if no infinite well-formed proof tree for $\mathscr{S}$ exists.*

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Infinite well-formed proof trees T have a *single* infinite branch [14, Remark 1] and [15, Section 2.4]:

$$\dfrac{\vec{T}_1 \quad \dfrac{\vdots}{G_1}(\rho_1) \quad \vec{O}_1}{G_0}(\rho_0) \qquad \dfrac{\vec{T}_n \quad \dfrac{\vdots}{G_n}(\rho_n) \quad \vec{O}_n}{G_1}$$

where for all $i \in \mathbb{N}$, a (possibly renamed) rule ρ_i of the form $\frac{B_1 \cdots B_{n_i}}{A_i}$ for some $n_i > 0$ has been applied, $\vec{T}_i$ are sequences of $m_i - 1$ closed proof trees for some $0 < m_i \leq n_i$, and $\vec{O}_i$ are sequences of $n_i - m_i$ open goals. For all $i \geq 0$, goal G_i is the root of the infinite subtree immediately above it. Formally, T is the limit of the sequence $\{S_i\}_{i\in\mathbb{N}}$, where $S_0 = G_0$ and for all $i \geq 0$, S_{i+1} is obtained from S_i by replacing the open goal G_i by $\frac{\vec{T}_{i+1} \quad G_{i+1} \quad \vec{O}_{i+1}}{G_i}$. Note that, for all $i \geq 0$, $S_i \subset S_{i+1}$, as required [15, Section 2.4].

3 Directions in operational termination

The following definition, which formalizes our previous observation about the existence of a single infinite branch in any infinite well-formed proof tree, improves a previous one in [15, Definition 7].

Definition 2 (Spine) *Let T be an infinite well-formed proof tree with a single infinite branch as above. We call the infinite sequence $(\langle \rho_i, m_i \rangle)_{i\in\mathbb{N}}$ the* spine *of T, denoted spine(T).*

A *proof jump* for $\mathscr{I}(\mathscr{S})$ (or just $\mathscr{S}$) is a pair $\psi = (A \Uparrow \vec{B}_n)$, where $n > 0$ and $A, B_1, \ldots, B_n \in Form(\mathscr{S})$; A and B_n are called the *head* and the *hook* of ψ, respectively; $B_1, \ldots, B_{n-1}$ is the *conditional part* [14]; let $Jumps(\mathscr{S})$ be the set of all such proof jumps. The set of *proof jumps of the inference rules in* $\mathscr{I}(\mathscr{S})$ is $\mathscr{J}_{\mathscr{S}} = \{A \Uparrow \vec{B}_i \mid \frac{\vec{B}_m}{A} \in \mathscr{I}(\mathscr{S}), 1 \leq i \leq m\}$. Given a rule $\frac{B_1 \cdots B_m}{A}$ with label ρ and $1 \leq i \leq m$, $[\rho]^i$ denotes its i-th proof jump $A \Uparrow B_1, \ldots, B_i$. Proof jumps $A \Uparrow \vec{B}_n$ provide a clear distinction between the *direction* of progress of an infinite behavior during a *computation-as-proof* (by means of the head A and hook B_n) and the logical context surrounding such a progress (we can assume $B_1, \ldots, B_{n-1}$ in the conditional part provable, after instantiation). We use proof jumps to track the spine of infinite well-formed proof trees (see Theorem 1 below). First, we consider some examples of proof jumps associated to CTRS theories.

Example 4 *Consider the following CTRS $\mathscr{R}$ [22, Example 7.2.45]*

$$\mathtt{a} \to \mathtt{a} \quad \text{if} \quad \mathtt{b} \to x, \mathtt{c} \to x \qquad (1)$$

$$\mathtt{b} \to \mathtt{d} \quad \text{if} \quad \mathtt{d} \to x, \mathtt{e} \to x \qquad (2)$$

$$\mathtt{c} \to \mathtt{d} \quad \text{if} \quad \mathtt{d} \to x, \mathtt{e} \to x \qquad (3)$$

where a, b, c, d, *and* e *are* constant symbols *and x is a variable. The inference system in Figure 1 specializes to this inference system $\mathscr{I}(\mathscr{R})$:*

$$(Rf)\ \frac{}{x \to^* x} \qquad (T)\ \frac{x \to y \quad y \to^* z}{x \to^* z}$$

$$(Rl)_{(1)}\ \frac{\mathtt{b} \to^* x \quad \mathtt{c} \to^* x}{\mathtt{a} \to \mathtt{a}} \qquad (Rl)_{(2)}\ \frac{\mathtt{d} \to^* x \quad \mathtt{e} \to^* x}{\mathtt{b} \to \mathtt{d}} \qquad (Rl)_{(3)}\ \frac{\mathtt{d} \to^* x \quad \mathtt{e} \to^* x}{\mathtt{c} \to \mathtt{d}}$$

where y and z are also variables. The proof jumps $\mathscr{J}_{\mathscr{R}}$ for $\mathscr{I}(\mathscr{R})$ are:

$[T]^1$ $\quad x \to^* z \Uparrow x \to y$ $\qquad$ $[Rl_{(1)}]^1$ $\mathtt{a} \to \mathtt{a} \Uparrow \mathtt{b} \to^* x$ $\qquad$ $[Rl_{(2)}]^1$ $\mathtt{b} \to \mathtt{d} \Uparrow \mathtt{d} \to^* x$

$[T]^2$ $\quad x \to^* z \Uparrow x \to y,\ y \to^* z$ $\qquad$ $[Rl_{(1)}]^2$ $\mathtt{a} \to \mathtt{a} \Uparrow \mathtt{b} \to^* x,\ \mathtt{c} \to^* x$ $\qquad$ $[Rl_{(2)}]^2$ $\mathtt{b} \to \mathtt{d} \Uparrow \mathtt{d} \to^* x,\ \mathtt{e} \to^* x$

$[Rl_{(3)}]^1$ $\mathtt{c} \to \mathtt{d} \Uparrow \mathtt{d} \to^* x$ $\qquad$ $[Rl_{(3)}]^2$ $\mathtt{c} \to \mathtt{d} \Uparrow \mathtt{d} \to^* x,\ \mathtt{e} \to^* x$

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Example 5 *For the following CTRS $\mathscr{R}$ [4, page 46]:*

$$\mathtt{a} \to \mathtt{b} \quad (4) \qquad \mathtt{f}(\mathtt{a}) \to \mathtt{b} \quad (5) \qquad \mathtt{g}(x) \to \mathtt{g}(\mathtt{a}) \text{ if } \mathtt{f}(x) \to x \quad (6)$$

we obtain the following inference system $\mathscr{I}(\mathscr{R})$ where x, y and z are variables:

$$(Rf)\ \frac{}{x \to^* x} \quad (T)\ \frac{x \to y \quad y \to^* z}{x \to^* z} \quad (C)_{\mathtt{f},1}\ \frac{x \to y}{\mathtt{f}(x) \to \mathtt{f}(y)} \quad (C)_{\mathtt{g},1}\ \frac{x \to y}{\mathtt{g}(x) \to \mathtt{g}(y)}$$

$$(Rl)_{(4)}\ \frac{}{\mathtt{a} \to \mathtt{b}} \quad (Rl)_{(5)}\ \frac{}{\mathtt{f}(\mathtt{a}) \to \mathtt{b}} \quad (Rl)_{(6)}\ \frac{\mathtt{f}(x) \to^* x}{\mathtt{g}(x) \to \mathtt{g}(\mathtt{a})}$$

The set $\mathscr{J}_{\mathscr{R}}$ of proof jumps for these inference rules consists of the following proof jumps:

$$[T]^1 \quad x \to^* z \Uparrow x \to y \qquad [T]^2 \quad x \to^* z \Uparrow x \to y,\ y \to^* z$$

$$[C_{\mathtt{f},1}]^1\ \mathtt{f}(x) \to \mathtt{f}(y) \Uparrow x \to y \quad [C_{\mathtt{g},1}]^1 \quad \mathtt{g}(x) \to \mathtt{g}(y) \Uparrow x \to y \qquad [Rl_{(6)}]^1\ \mathtt{g}(x) \to \mathtt{g}(\mathtt{a}) \Uparrow \mathtt{f}(x) \to^* x$$

Proof jumps are used to track the spine of infinite well-formed proof trees as *OT chains*: an $(\mathscr{S}, \mathscr{J})$-*chain* (generically called *OT chain*) is a sequence $(\psi_i)_{i\geq 1}$ of proof jumps $\psi_i : (A^i \Uparrow \vec{B}^i_{n_i}) \in \mathscr{J}$ together with a substitution σ such that for all $i \geq 1$, $\sigma(B^i_{n_i}) = \sigma(A^{i+1})$ and for all j, $1 \leq j < n_i$, $\mathscr{S} \vdash \sigma(B^i_j)$ [14].

Theorem 1 [14] *A theory $\mathscr{S}$ in a logic $\mathscr{L}$ is* operationally terminating *if and only if there is no* infinite $(\mathscr{S}, \mathscr{J}_{\mathscr{S}})$*-chain.*

Given a set $\mathscr{J}$ of proof jumps, denote as $\mathfrak{H}(\mathscr{J}) = \{A \mid A \Uparrow \vec{B}_n \in \mathscr{J}\}$ (resp. $\mathfrak{K}(\mathscr{J}) = \{B_n \mid A \Uparrow \vec{B}_n \in \mathscr{J}\}$) the set of *heads* (resp. *hooks*) of proof jumps in $\mathscr{J}$. Let $\mathscr{H}_{\mathscr{S}} = \mathfrak{H}(\mathscr{J}_{\mathscr{S}})$ and $\mathscr{K}_{\mathscr{S}} = \mathfrak{K}(\mathscr{J}_{\mathscr{S}})$.

Example 6 *For $\mathscr{J}_{\mathscr{R}}$ in Example 4, we have $\mathscr{H}_{\mathscr{R}} = \{x \to^* z, \mathtt{a} \to \mathtt{a}, \mathtt{b} \to \mathtt{d}, \mathtt{c} \to \mathtt{d}\}$ and $\mathscr{K}_{\mathscr{R}} = \{x \to y, y \to^* z, \mathtt{b} \to^* x, \mathtt{c} \to^* x, \mathtt{d} \to^* x, \mathtt{e} \to^* x\}$.*

We write $\varphi \leq \varphi'$ (φ subsumes φ') iff $\sigma(\varphi) = \varphi'$ for some substitution σ [23]. For sets $\Phi, \Phi' \subseteq Form(\mathscr{S})$, write $\Phi \leq \Phi'$ (Φ subsumes Φ') iff for all $\varphi' \in \Phi'$, there is $\varphi \in \Phi$ such that $\varphi \leq \varphi'$, i.e., every formula in Φ' is subsumed by at least one formula in Φ. We write $\Phi \trianglelefteq \Phi'$ (Φ *tightly* subsumes Φ') iff $\Phi \leq \Phi'$ and for all $\varphi \in \Phi$ there is $\varphi' \in \Phi'$ such that $\varphi \leq \varphi'$ (each formula in Φ subsumes at least one formula in Φ'); $\leq$ and $\trianglelefteq$ are quasi-orderings (reflexive and transitive). If $\Phi \leq \Phi'$ we can extract a subset $\Phi_0 \subseteq \Phi$ satisfying $\Phi_0 \trianglelefteq \Phi'$ by removing unnecessary formulas from Φ. We also write $\Phi \doteq^{\circ} \Phi'$ (Φ and Φ' tightly unify) iff (i) for all $\phi' \in \Phi'$ there is $\phi \in \Phi$ such that ϕ and ϕ' unify (written $\phi \doteq \phi'$) and (ii) for all $\phi \in \Phi$ there is $\phi' \in \Phi'$ such that $\phi \doteq \phi'$. A *direction* for $\mathscr{I}(\mathscr{S})$ is a set Δ of formulas that tightly subsumes the heads of a nonempty subset of proof jumps for $\mathscr{I}(\mathscr{S})$ and which tightly unifies with the hooks.

Definition 3 (Direction) *A set $\Delta \subseteq Form(\mathscr{S})$ of formulas is a* direction *for $\mathscr{I}(\mathscr{S})$ (or $\mathscr{S}$) iff $\Delta \trianglelefteq \mathfrak{H}(\mathscr{J})$ and $\Delta \doteq^{\circ} \mathfrak{K}(\mathscr{J})$ for some nonempty set $\mathscr{J} \subseteq \mathscr{J}_{\mathscr{S}}$.*

Directions Δ *classify* infinite well-formed proof trees as Δ*-directed trees*.

Definition 4 (Directed tree) *An infinite well-formed proof tree T with $spine(T) = (\langle \rho_i, m_i \rangle)_{i\in\mathbb{N}}$ is Δ-directed (or has direction Δ) if there is $N \in \mathbb{N}$ such that*

1. *$\Delta \trianglelefteq \mathfrak{H}(\{[\rho_i]^{m_i} \mid i \geq N\})$, i.e., Δ subsumes all* heads *of proof jumps after the Nth.*
2. *For all $\delta \in \Delta$, there is an infinite set $I_\psi \subseteq \mathbb{N}$ such that for all $i \in I_\psi$, with $[\rho_i]^{m_i} = A \Uparrow \vec{B}_{m_i}$, we have $\delta \leq A$ i.e., all formulas in Δ subsume infinitely many proof jumps from $spine(T)$.*

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Although subsumption ($\leq$) is used in item 1 of Definition 4, item 2 actually implies the *tight* subsumption $\trianglelefteq$. Furthermore, directions and *hooks* of the proof jumps $[\rho_i]^{m_i}$ in the tail of $spine(T)$ after N are also related: since there is a substitution σ (the one which is used to develop the proof tree) satisfying $\sigma(B^i_{n_i}) = \sigma(A^{i+1})$ for all $i \in \mathbb{N}$ (and appropriate renamings of rules), and $A^{i+1} = \theta_{i+1}(\delta_{i+1})$ for all $i \geq N$ (for some $\delta_{i+1} \in \Delta$ and substitution θ_{i+1}), we have that $B^i_{n_i}$ and δ *unify*, i.e., there is a substitution ς_i such that $\varsigma_i(B^i_{n_i}) = \varsigma_i(\delta)$. This justifies Definition 3.

Example 7 *For $\mathscr{R}$ in Example 4, with $\mathscr{H}_{\mathscr{R}}$ and $\mathscr{K}_{\mathscr{R}}$ in Example 6, we have $\mathscr{H}_{\mathscr{R}} \doteq \mathscr{K}_{\mathscr{R}}$. Thus, $\mathscr{H}_{\mathscr{R}}$ is a direction for $\mathscr{I}(\mathscr{R})$. However, $\{x \to y\}$ is* not*: the proof jumps $[Rl_{(1)}]^j$, $[Rl_{(2)}]^j$, and $[Rl_{(3)}]^j$ for $j \in \{1,2\}$ are all and the only proof jumps whose heads are subsumed by $x \to y$, but no hook unifies with it. Similarly, $\{\mathtt{a} \to \mathtt{a}\} \subseteq \mathscr{H}_{\mathscr{R}}$ is* not *a direction for $\mathscr{I}(\mathscr{R})$ either. In contrast, $\{x \to y\}$ is a direction for $\mathscr{R}$ in Example 5 due to the proof jumps $[C_{\mathtt{f},1}]^1$ and $[C_{\mathtt{g},1}]^1$.*

Proposition 1 *Let Δ and Δ' be directions for $\mathscr{S}$ such that $\Delta' \trianglelefteq \Delta$. Every Δ-directed tree is Δ'-directed.*

Definition 5 (Directed operational termination) *Let $\mathscr{S}$ be a theory and Δ be a direction for $\mathscr{S}$. We say that $\mathscr{S}$ is Δ-directed operationally terminating (Δ-DOT for short) if there is no Δ-directed proof tree.*

As a simple consequence of Proposition 1, we have the following.

Proposition 2 *Let Δ and Δ' be directions such that $\Delta' \trianglelefteq \Delta$. If $\mathscr{S}$ is Δ'-DOT, then $\mathscr{S}$ is Δ-DOT.*

Operational termination and directed operational termination are related.

Theorem 2 *A theory $\mathscr{S}$ with $\mathscr{J}_{\mathscr{S}}$ finite is operationally terminating iff $\mathscr{S}$ is Δ-DOT for all directions Δ such that $\Delta \subseteq \mathscr{H}_{\mathscr{S}}$.*

Not every subset of $\mathscr{H}_{\mathscr{S}}$ is a direction for $\mathscr{S}$ (see Example 7) but this could be the case for some theories; this would lead to a big (exponential!) number of directions. Furthermore, operational termination and $\mathscr{H}_{\mathscr{S}}$-DOT are *not* necessarily equivalent (see Example 10 below). We investigate smaller sets of directions to be used in proofs of operational termination.

Canonical directions.

Given a set of formulas $\Phi \subseteq Form(\mathscr{S})$, the existence of a *least* set $\Phi^{\downarrow} \subseteq Form(\mathscr{S})$ such that $\Phi^{\downarrow} \trianglelefteq \Phi$ may depend on the formulas in Φ. For instance, in first-order logic $\Phi^{\downarrow}$ exists (up to renaming) for sets Φ of *atomic* formulas: $\Phi^{\downarrow}$ is the set of atoms $P(x_1,\ldots,x_n)$ with distinct variables $x_1,\ldots,x_n$ for each n-ary predicate symbol P occurring in atoms in Φ (see Example 8 below).

Example 8 *For $\mathscr{H}_{\mathscr{R}}$ in Example 6, we have $\mathscr{H}^{\downarrow}_{\mathscr{R}} = \{x \to y, x \to^* y\}$, where x and y are* distinct *variables.*

In the following, when considering a set of formulas $\Phi \subseteq Form(\mathscr{S})$ we assume the existence of a *least* set $\Phi^{\downarrow} \subseteq Form(\mathscr{S})$ such that $\Phi^{\downarrow} \trianglelefteq \Phi$. In particular, every direction $\Delta \subseteq \mathscr{H}^{\downarrow}_{\mathscr{S}}$ for $\mathscr{S}$ is called a *canonical* direction. We have the following.

Theorem 3 *A theory $\mathscr{S}$ in a logic $\mathscr{L}$ with $\mathscr{H}_{\mathscr{S}}$ finite is operationally terminating iff $\mathscr{S}$ is Δ-DOT for all canonical directions $\Delta^{\downarrow} \subseteq \mathscr{H}^{\downarrow}_{\mathscr{S}}$.*

4 Directed OT and termination properties

In this section we show a number of cases where the notion of directed operational termination can be used to characterize other termination-related notions. For simplicity, we often restrict the attention to (C)TRS logic (Figure 1) in specific examples.

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

4.1 Operational termination of CTRSs

A CTRS $\mathscr{R}$ is operationally terminating if there is no infinite well-formed proof tree for $\mathscr{I}(\mathscr{R})$ obtained from the inference system in Figure 1 (see Example 3). In [15], a characterization of operational termination of CTRSs as the *conjunction* of two termination properties has been given. The first property is *termination*, i.e., the absence of infinite rewrite sequences, which was proved equivalent to $\{x \to^* y\}$-DOT as defined above (see [15, Proposition 12]). The second property was coined *V-termination* and defined as the absence of infinite well-formed proof trees T whose $spine(T)$ contains infinitely many uses of rules ρ of the form $\frac{B_1 \cdots B_n}{s \to t}$ [15, Definition 13] where s and t are terms. This property corresponds to $\{x \to y, x \to^* y\}$-DOT due to Theorem 3 and the following result.

Theorem 4 *Every CTRS $\mathscr{R}$ is $\{x \to y\}$-DOT.*

Therefore, the 'bidimensional' characterization of operational termination in [15, Theorem 14] can be rephrased as follows (and follows from Theorem 3 and 4).

Theorem 5 *A CTRS is operationally terminating if and only if it is both $\{x \to^* y\}$-DOT and $\{x \to y, x \to^* y\}$-DOT.*

4.2 Well-foundedness and operational termination

Let $\Omega = (S, \Sigma, \Pi)$ be a many-sorted signature with predicates [5] and $\pi \in \Pi_{ss}$ be a binary predicate for some sort $s \in S$. Let $\mathscr{L}_\Omega$ be a logic whose set of formulas $Form(\mathscr{L}_\Omega)$ consists of the first-order formulas obtained from Ω. Given a theory $\mathscr{S} \in Th(\mathscr{L}_\Omega)$, we can define a binary relation $R_\pi \subseteq \mathscr{T}_\Sigma(\mathscr{X})_s \times \mathscr{T}_\Sigma(\mathscr{X})_s$ on sorted terms as follows: for all $t, u \in \mathscr{T}_\Sigma(\mathscr{X})_s$, $t \; R_\pi \; u \Leftrightarrow \mathscr{S} \vdash \pi(t,u)$. The following result shows that well-foundedness of such binary relations can be captured by directed operational termination after an appropriate *extension* of the logic.

Theorem 6 *Let $\Omega = (S, \Sigma, \Pi)$, $\mathscr{S} \subseteq Form(\mathscr{L}_\Omega)$, $\pi \in \Pi_{ss}$ for some $s \in S$, seq_π be a new predicate symbol, and $(\rho_\pi) \frac{\pi(x,y) \quad \mathsf{seq}_\pi(y)}{\mathsf{seq}_\pi(x)}$ be a new inference rule. Then, R_π is well-founded iff $\mathscr{I}(\mathscr{S}) \cup \{\rho_\pi\}$ is $\{\mathsf{seq}_\pi(x)\}$-DOT.*

Rule (ρ_π) creates a *sequence* to investigate well-foundedness of R_π.

4.2.1 Rewriting logic

Consider Meseguer's *Rewriting Logic* [18]. According to [20], a *rewrite theory* $\mathscr{R} = (\Sigma, E, R)$ consists of an equational theory (Σ, E) with function symbols Σ, equations E, and a set R of labeled rules $\alpha : \ell \to r$. The inference system $\mathscr{I}_{RL}(\mathscr{R})$ for *rewriting logic* in Figure 2 borrows [20, Section 3.1]. There is a rule $(C)_{f,i}$ for each k-ary function symbol $f \in \Sigma$ and $1 \le i \le k$. There are rules $(Rl)_\alpha$ and $(Rp)_\alpha$ for each $\alpha : \ell \to r \in R$ with $Vars(\ell) \cup Vars(r) = \{x_1, \ldots, x_n\}$ and r' is r with all occurrences of variables x_i replaced by y_i for all $1 \le i \le n$. Symbol '$=$' in rule *(Eq)* is equality *modulo a set of equations E*. This involves an inference system $\mathscr{I}_{EL}(E)$ for equational logic, cf. [8, Table 6.1], which is not explicit in [20] but also plays a role when modeling operational termination of $\mathscr{R}$:

$$(E_=)_{s,t} \; \frac{}{s = t} \qquad (Rf_=) \; \frac{}{x = x} \qquad (S_=) \; \frac{y = x}{x = y}$$

$$(T_=) \; \frac{x = y \quad y = z}{x = z} \qquad (C_=)_f \; \frac{x_1 = y_1 \quad \cdots \quad x_k = y_k}{f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k)}$$

$$(Eq)\ \frac{x \to y \quad x = x' \quad y = y'}{x' \to y'} \qquad (C)_{f,i}\ \frac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to f(x_1,\ldots,y_i,\ldots,x_k)}$$

$$(T)\ \frac{x \to y \quad y \to z}{x \to z} \qquad (Rl)_\alpha\ \frac{}{\ell \to r} \qquad (Rp)_\alpha\ \frac{x_1 \to y_1 \quad \cdots \quad x_n \to y_n}{\ell \to r'}$$

Figure 2: Schematic inference rules $\mathscr{I}_{RL}(\mathscr{R})$ for (unconditional) rewriting logic (cf. [20])

$$(Rl_i)\ \frac{}{\ell \to_i r} \text{ where } \ell \to r \in \mathscr{R}_i \qquad (Rf_i)\ \frac{}{x \to_i^* x}$$

$$(C_i)_{f,j}\ \frac{x_j \to_i y_j}{f(x_1,\ldots,x_j,\ldots,x_n) \to_i f(x_1,\ldots,y_j,\ldots,x_n)} \text{ where } f \in \mathscr{F}_1 \cup \mathscr{F}_2 \text{ and } 1 \le j \le ar(f)$$

$$(T_i)\ \frac{x \to_i y \quad y \to_i^* z}{x \to_i^* z} \qquad (R_{21})\ \frac{x \to_2^* y \quad y \to_1 z}{x \to_{21} z}$$

Figure 3: Inference System for termination of $\mathscr{R}_1$ relative to $\mathscr{R}_2$ (with $i \in \{1,2\}$)

with rules $(E_=)_{s,t}$ for each $(s,t) \in E$ and rules $(C_=)_f$ for each $f \in \Sigma$. Operational termination *fails* to hold in $\mathscr{I}_{RL}(\mathscr{R})$. For instance, by a repeated use of rule (T) we obtain the infinite proof tree

$$\dfrac{\dfrac{\dfrac{\vdots}{s \to s_2}(T) \qquad s_2 \to s_1}{s \to s_1}(T) \qquad s_1 \to t}{s \to t}(T)$$

where s, $s_1,\ldots$ and t are arbitrary terms. Rules $(S_=)$ and $(T_=)$ in $\mathscr{I}_{EL}(E)$ have similar problems. As in Example 2, there is a 'mismatch' between operational termination of $\mathscr{I}_{RL}(\mathscr{R}) \cup \mathscr{I}_{EL}(E)$ (which *never* holds, disregarding the considered rewrite theory $\mathscr{R}$!) and *well-foundedness* of the many-step rewriting modulo E relation $s \to^+_{\mathscr{R}/E}$ defined as follows: for all $s,t \in \mathscr{T}_\Sigma(\mathscr{X})$, $s \to^+_{\mathscr{R}/E} t \Leftrightarrow \mathscr{I}_{RL}(\mathscr{R}) \cup \mathscr{I}_{EL}(E) \vdash s \to t$ (which depends on $\mathscr{R}$ and E). By Theorem 6, though, we know that well-foundedness of $\to^+_{\mathscr{R}/E}$ is equivalent to $\{\mathsf{seq}_\to(x)\}$-DOT of $\mathscr{I}_{RL}(\mathscr{R}) \cup \mathscr{I}_{EL}(E) \cup \{\rho_\to\}$, where $\rho_\to$ is $\frac{x \to y \quad \mathsf{seq}_\to(y)}{\mathsf{seq}_\to(x)}$.

4.2.2 Relative termination

Given binary relations $\to_1, \to_2 \subseteq A \times A$ on a set A, $\to_1$ is called *relatively terminating* with respect to $\to_2$ if every infinite $\to_1 \cup \to_2$-sequence contains only finitely many $\to_1$-steps. Geser has proved that $\to_1$ is relatively terminating with respect to $\to_2$ iff $\to_2^* \circ \to_1$ is well-founded [3]. Given TRSs $\mathscr{R}_1$ and $\mathscr{R}_2$ over signatures $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively, termination of $\mathscr{R}_1$ relative to $\mathscr{R}_2$ can be investigated as well-foundedness of $\to^*_{\mathscr{R}_2} \circ \to_{\mathscr{R}_1}$ (over $\mathscr{T}(\mathscr{F}_1 \cup \mathscr{F}_2, \mathscr{X})$). The inference system $\mathscr{I}_{21}(\mathscr{R}_1,\mathscr{R}_2)$ in Figure 3 can be used to define $\to^*_{\mathscr{R}_2} \circ \to_{\mathscr{R}_1}$ as follows: $s \to^*_{\mathscr{R}_2} \circ \to_{\mathscr{R}_1} t$ iff $\mathscr{I}_{21}(\mathscr{R}_1,\mathscr{R}_2) \vdash s \to_{21} t$. Termination of $\mathscr{R}_1$ relative to $\mathscr{R}_2$ can *not* be characterized as the operational termination of $\mathscr{I}_{21}(\mathscr{R}_1,\mathscr{R}_2)$: if $\mathscr{R}_2$ is *not* terminating, then due to (Rl_2), (C_2), and (T_2), $\mathscr{I}_{21}(\mathscr{R}_1,\mathscr{R}_2)$ is not operationally terminating. By Theorem 6, termination of $\mathscr{R}_1$ relative to $\mathscr{R}_2$ is $\{\mathsf{seq}_{\to_{21}}(x)\}$-DOT of $\mathscr{I}_{21}(\mathscr{R}_1,\mathscr{R}_2) \cup \{\rho_{\to_{21}}\}$.

4.3 Termination with dependency pairs

In this section we establish a connection between directed operational termination and the termination analysis of TRSs using dependency pairs [1]. The crucial observation is that the analysis of infinite

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

$$(R)_\alpha\ \frac{}{\ell \to r} \quad \text{where } \alpha : \ell \to_{\mathscr{R}} r \in \mathscr{R}$$

$$(\mathrm{C})_{f,i}\ \frac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_n) \to f(x_1,\ldots,y_i,\ldots,x_n)} \quad \text{where } f \in \mathscr{F} \text{ and } 1 \le i \le ar(f)$$

$$(Rf_{>\Lambda})\ \frac{}{x \xrightarrow{>\Lambda}{}^* x}$$

$$(C_{>\Lambda})_{f,i}\ \frac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_n) \xrightarrow{>\Lambda} f(x_1,\ldots,y_i,\ldots,x_n)} \quad \text{where } f \in \mathscr{F} \text{ and } 1 \le i \le ar(f)$$

$$(T_{>\Lambda})\ \frac{x \xrightarrow{>\Lambda} y \qquad y \xrightarrow{>\Lambda}{}^* z}{x \xrightarrow{>\Lambda}{}^* z}$$

$$(\mathrm{M})_{\alpha,p}\ \frac{x \xrightarrow{>\Lambda}{}^* \ell \qquad \mathsf{m}(r|_p)}{\mathsf{m}(x)} \quad \text{where } \alpha : \ell \to r \in \mathscr{R} \text{ and } root(r|_p) \in \mathscr{D}$$

Figure 4: Inference System approximating minimality of terms in TRSs

(i.e., nonterminating) rewrite sequences can be restricted to those starting from a *minimal* term. Given a TRS $\mathscr{R}$, a nonterminating term is *minimal* iff its proper subterms are terminating; $\mathscr{T}_\infty$ is the set of minimal nonterminating terms associated to $\mathscr{R}$ [7]. Every nonterminating term s contains a minimal nonterminating term $t \in \mathscr{T}_\infty$ and minimal nonterminating terms t are always rooted by a *defined* symbol $f \in \mathscr{D}$, i.e., a symbol occurring at the topmost position of the left-hand side ℓ of a rule $\ell \to r \in \mathscr{R}$. As discussed in [7], considering the structure of the infinite rewrite sequences starting from a minimal nonterminating term $t \in \mathscr{T}_\infty$ can be helpful to come to the notion of dependency pair [1]. Such sequences, of the form $t \xrightarrow{>\Lambda}{}^* \sigma(\ell) \xrightarrow{\Lambda} \sigma(r) \trianglerighteq u$ (where $\xrightarrow{\Lambda}$ is rewriting restricted to the *topmost* position Λ, $\xrightarrow{>\Lambda}{}^*$ is many step rewriting *below* the root, t and u are terms, $\ell \to r$ is a rewrite rule, and $\subseteq$ is the *subterm* relation), are considered in [7]. Their role is formalized in the following result.

Proposition 3 [7, Lemma 1] *Let $\mathscr{R} = (\mathscr{C} \uplus \mathscr{D}, R)$ be a TRS. For all $t \in \mathscr{T}_\infty$, there exist $\ell \to r \in R$, a substitution σ and a term $u \in \mathscr{T}_\infty$ such that $root(u) \in \mathscr{D}$, $t \xrightarrow{>\Lambda}{}^* \sigma(\ell) \xrightarrow{\Lambda} \sigma(r) \trianglerighteq u$, and there is a nonvariable subterm v of r, $r \trianglerighteq v$, such that $u = \sigma(v)$.*

Proposition 3 can be written *in logic form* by the inference system $\mathscr{I}_M(\mathscr{R})$ in Figure 4 (where m is a new, monadic, predicate symbol). We have the following consequence of Proposition 3.

Corollary 1 *A TRS $\mathscr{R}$ is terminating if and only if $\mathscr{I}_M(\mathscr{R})$ is $\{\mathsf{m}(x)\}$-DOT.*

5 Directed OT chains and directed proof graphs

The following definition distinguishes the kind of OT-chains that can be used to 'track' the (tail of the) spine of a Δ-directed tree.

Definition 6 (Directed OT chain) *Given a direction Δ, an infinite $(\mathscr{S}, \mathscr{J})$-chain $(A^i \Uparrow \vec{B}^i_{n_i})_{i\in\mathbb{N}}$ is said Δ-directed iff $\Delta \trianglelefteq \mathfrak{H}(\mathscr{J})$ and $\Delta \doteq \mathfrak{K}(\mathscr{J})$ for $\mathscr{J} = \{A^i \Uparrow \vec{B}^i_{n_i} \mid i \in \mathbb{N}\}$.*

OT chains are intended to capture the *spine* of an infinite well-formed proof tree. Actually, the spine $(\langle \rho_i, m_i \rangle)_{i\in\mathbb{N}}$ of an infinite well-formed proof tree T for $\mathscr{S}$ (see Definition 2) can be seen as an OT chain $([\rho_i]^{m_i})_{i\in\mathbb{N}}$. Similarly, Δ-directed OT chains capture the spine of Δ-directed trees, which can be seen as a Δ-directed OT chain. In the following, $\mathscr{J}\lfloor_\Delta$ is the biggest subset of $\mathscr{J}$ such that $\Delta \trianglelefteq \mathfrak{H}(\mathscr{J}\lfloor_\Delta)$ and $\Delta \doteq \mathfrak{K}(\mathscr{J}\lfloor_\Delta)$ (if there is such a subset), or $\mathscr{J}\lfloor_\Delta = \emptyset$ otherwise.

Example 9 *For $\mathscr{R}$ and $\mathscr{J}_\mathscr{R}$ in Example 4: $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to y\}} = \emptyset$, $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to^* y\}} = \{[T]^2\}$, and $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to y, x\to^* y\}} = \mathscr{J}_\mathscr{R}$. For $\mathscr{R}$ and $\mathscr{J}_\mathscr{R}$ in Example 5, we have: $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to y\}} = \{[C_{\mathtt{f},1}]^1], [C_{\mathtt{g},1}]^1]\}$, $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to^* y\}} = \{[T]^2\}$, and $\mathscr{J}_\mathscr{R}\lfloor_{\{x\to y, x\to^* y\}} = \mathscr{J}_\mathscr{R}$.*

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Theorem 7 *A theory $\mathscr{S}$ is* Δ-DOT *iff there is no* Δ*-directed* $(\mathscr{S}, \mathscr{J}_{\mathscr{S}}\lfloor_\Delta)$*-chain.*

Now we show that (as claimed after Theorem 2) operational termination and $\mathscr{H}_{\mathscr{S}}$-DOT do *not* always coincide.

Example 10 *If rule (1) in the CTRS $\mathscr{R}$ of Example 4 is slightly changed into*

$$\mathtt{a} \to \mathtt{a} \quad \text{if} \quad \mathtt{a} \to x, \mathtt{c} \to x \tag{7}$$

then, for $\mathscr{R}' = \{(7),(2),(3)\}$ we have $\mathscr{H}_{\mathscr{R}'} = \mathscr{H}_{\mathscr{R}} = \{x \to^* z, \mathtt{a} \to \mathtt{a}, \mathtt{b} \to \mathtt{d}, \mathtt{c} \to \mathtt{d}\}$ (see Example 6) and $\mathscr{K}_{\mathscr{R}'} = \{x \to y, y \to^* z, \mathtt{a} \to^* x, \mathtt{c} \to^* x, \mathtt{d} \to^* x, \mathtt{e} \to^* x\}$. Thus, $\mathscr{H}_{\mathscr{R}'}$ is a direction for $\mathscr{R}'$. Although $\mathscr{R}'$ is $\mathscr{H}_{\mathscr{R}'}$-OT, it is not *operationally terminating due to the following $(\mathscr{R}', \mathscr{J}_{\mathscr{R}'})$-chain:*

$$(\mathtt{a} \to \mathtt{a} \Uparrow \mathtt{a} \to^* x_i, \quad x'_i \to^* z'_i \Uparrow x'_i \to y'_i)_{i \geq 1}$$

which uses the new proof jump $[Rl_{(7)}]^1$ and $[T]^1$ with $\sigma(x_i) = \sigma(x'_i) = \sigma(y'_i) = \sigma(z'_i) = \mathtt{a}$ for all $i \geq 1$. This also shows that $\mathscr{R}'$ is not $\{\mathtt{a} \to \mathtt{a}, x \to^* z\}$-DOT (note that $\{\mathtt{a} \to \mathtt{a}, x \to^* z\}$ is a direction for $\mathscr{R}'$).

Definition 7 (Feasibility/Infeasibility) *A formula $\varphi \in Form(\mathscr{S})$ is* $\mathscr{S}$-feasible *(or just* feasible *if no confusion arises) if there is a substitution σ such that $\mathscr{S} \vdash \sigma(\varphi)$; otherwise, we call it* infeasible. *An inference rule $(\rho)\ \frac{B_1 \cdots B_n}{A}$ is* feasible *if $B_1 \wedge \cdots \wedge B_n$ is feasible, i.e., there is an instantiation of the formulas in the conditional part of the rule which makes them simultaneously provable, thus enabling the use of the inference rule in a proof of $\sigma(A)$; we call ρ infeasible otherwise. Also, $A \Uparrow \vec{B}_n$ is (in)feasible if $B_1 \wedge \cdots \wedge B_{n-1}$ is (in)feasible.*

(In)feasibility is, in general, undecidable, but some methods to prove *infeasibility* of first-order formulas have been investigated in [9]. In this paper, we will try to prove infeasibility of formulas by using existing methods (e.g., [9]); otherwise, we will consider them feasible.

Example 11 *Rule $(Rl)_{(6)}$ for $\mathscr{R}$ in Example 5 is* infeasible *(cf. [9, Example 9]).*

Let $\mathscr{S}^+$ (resp. $\mathscr{J}^+$) be (an overapproximation to) the feasible subset of $\mathscr{S}$ (resp. $\mathscr{J}$).

Corollary 2 *A theory $\mathscr{S}$ is* Δ-DOT *iff there is no* Δ*-directed* $(\mathscr{S}^+, \mathscr{J}^+_{\mathscr{S}}\lfloor_\Delta)$*-chain.*

In Corollary 2, computing the proof jumps $\mathscr{J}_{\mathscr{S}}$ of the inference rules in $\mathscr{I}(\mathscr{S})$, and then considering the feasible ones $\mathscr{J}^+_{\mathscr{S}}$ rather than computing the proof jumps $\mathscr{J}_{\mathscr{S}^+}$ of feasible rules $\mathscr{S}^+$ is essential: *infeasible* inference rules in $\mathscr{I}(\mathscr{S})$ may induce operational nontermination which would *not* be captured by $\mathscr{J}_{\mathscr{S}^+}$.

Example 12 *For the one-rule CTRS $\mathscr{R} = \{\mathtt{a} \to \mathtt{a} \text{ if } \mathtt{a} \to \mathtt{b}\}$, $\mathscr{I}(\mathscr{R})$ is*

$$(Rf)\ \frac{}{x \to^* x} \qquad (T)\ \frac{x \to y \quad y \to^* z}{x \to^* z} \qquad (Rl)\ \frac{\mathtt{a} \to^* \mathtt{b}}{\mathtt{a} \to \mathtt{a}}$$

Rule (Rl) is infeasible. *After removing it from $\mathscr{I}(\mathscr{R})$, the only proof jumps are $[T]^1$ and $[T]^2$. But $[T]^2$ is now infeasible (no inference rule applies to any instance of $x \to y$) and $[T]^1$ induces no chain due to the different predicates $\to^*$ and $\to$ in head and hook. There is no $(\mathscr{R}^+, \mathscr{J}_{\mathscr{R}^+})$-chain, but we cannot conclude operational termination of $\mathscr{R}$: (Rl) defines $[Rl]^1$, i.e., $\mathtt{a} \to \mathtt{a} \Uparrow \mathtt{a} \to^* \mathtt{b}$ and an infinite $(\mathscr{R}^+, \mathscr{J}_{\mathscr{R}})$-chain*

$$(\mathtt{a} \to \mathtt{a} \Uparrow \mathtt{a} \to^* \mathtt{b}, \quad x_i \to^* z_i \Uparrow x_i \to y_i)_{i \geq 1}$$

with substitution $\sigma(x_i) = \sigma(y_i) = \mathtt{a}$ and $\sigma(z_i) = \mathtt{b}$ for all $i \geq 1$.

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

Whenever $\mathscr{J}$ is finite, we can trace Δ-directed $(\mathscr{S},\mathscr{J})$-chains as *cycles* in the following *directed proof graph*:

Definition 8 (Directed proof graph) *Let $\mathscr{J} \subseteq Jumps(\mathscr{S})$ and Δ be a direction for $\mathscr{S}$. The set of* nodes *of the* Δ-directed proof graph $\mathsf{DPG}(\mathscr{S},\mathscr{J},\Delta)$ *is* $(\mathscr{J}\downarrow_\Delta)^+$. *There is an* arc *from $A \Uparrow \vec{B}_m$ to (a renamed version of) $A' \Uparrow \vec{B}'_n$ iff there is a substitution σ such that (i) $\sigma(B_m) = \sigma(A')$, (ii) $\sigma(B_1)\wedge\cdots\wedge\sigma(B_{m-1})\wedge\sigma(B'_1)\wedge\cdots\wedge\sigma(B'_{n-1})$ is* feasible*, and (iii) there is $A'' \Uparrow \vec{B}''_p \in \mathscr{J}^+$ such that $\sigma(B'_n)$ and A'' unify.*

Condition (ii) in Definition 8 ensures that goals $\sigma(B_j)$, $1 \le j < m$ and $\sigma(B'_j)$ for all $1 \le j < n$ can be proved, possibly by some further but *common* instantiation of the variables; also (iii) ensures that, after using two proof jumps to track a branch, more proof jumps could be used to continue into an *infinite* OT-chain. In this way, we may lose endpoints of finite chains due to the lack of some arcs, but this is not important for the purpose of capturing directed OT-chains. In the following, by a *cycle* in a graph G we mean a *sequence* $n_0,\ldots,n_m$ of nodes in G such that, $m > 0$ and for all i, $0 \le i < m$ there is an arc from n_i to n_{i+1} and $n_m = n_0$. We write cycles as *sets* $\{n_0,\ldots,n_{m-1}\}$ which actually may represent several cycles involving those nodes. Thus, in our setting, cycles $\mathscr{C}$ are treated as sets of proof jumps.

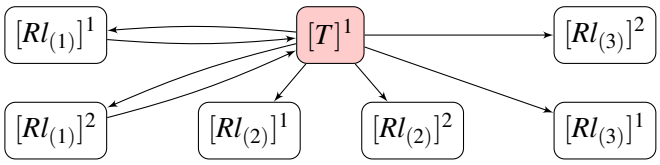
Theorem 8 *Let $\mathscr{S}$ be such that $\mathscr{J}_\mathscr{S}$ is finite and Δ be a direction. $\mathscr{S}$ is Δ-DOT if $\mathsf{DPG}(\mathscr{S},\mathscr{J}_\mathscr{S},\Delta)$ contains no cycle.*

In general, $\mathsf{DPG}(\mathscr{S},\mathscr{J},\Delta)$ is not computable. As remarked above, we (correctly) overapproximate as feasible those proof jumps which cannot be proved infeasible. Unification can also be undecidable for some logics. In the following definition we assume a unification algorithm available.

Definition 9 *The nodes of the* estimated directed proof graph $\mathsf{EDPG}(\mathscr{S},\mathscr{J},\Delta)$ *are an overapproximation of $(\mathscr{J}\downarrow_\Delta)^+$. There is an arc from $\psi : (A \Uparrow \vec{B}_m)$ to $\psi' : (A' \Uparrow \vec{B}'_n)$ iff there is $\sigma \in Sub(\mathscr{S})$ such that $\sigma(B_m) = \sigma(A')$, $\sigma(B_1)\wedge\cdots\wedge\sigma(B_{m-1})\wedge\sigma(B'_1)\wedge\cdots\wedge\sigma(B'_{n-1})$ is not proved infeasible, and $\sigma(B'_n)$ unify with A'' for some other proof jump $A'' \Uparrow \vec{B}''_p$.*

Example 13 *The estimated graphs for the proof jumps for $\mathscr{R}$ in Example 4 are:*

- $\mathsf{EDPG}(\mathscr{R},\mathscr{J}_\mathscr{R}\downarrow_{\{x\to y\}},\{x\to y\})$*, which has no node ($\mathscr{J}_\mathscr{R}\downarrow_{\{x\to y\}} = \emptyset$, see Example 9).*
- $\mathsf{EDPG}(\mathscr{R},\mathscr{J}_\mathscr{R}\downarrow_{\{x\to^* y\}},\{x\to^* y\})$*, which has no node: $\mathscr{J}_\mathscr{R}\downarrow_{\{x\to^* y\}}$ consists of a single node $[T]^2$ (see Example 9), but $(\mathscr{J}_\mathscr{R}\downarrow_{\{x\to^* y\}})^+ = \emptyset$ because the goal $x \to y$ in the conditional part of $[T]^2$ is infeasible*[1].
- $\mathsf{EDPG}(\mathscr{R},\mathscr{J}_\mathscr{R}\downarrow_{\{x\to y,x\to^* y\}},\{x\to y,x\to^* y\})$ *for $\mathscr{J}_\mathscr{R}\downarrow_{\{x\to y,x\to^* y\}} = \mathscr{J}_\mathscr{R}$ in Example 4 is:*



*where different colouring distinguishes the nodes that are covered by different formulas in $\mathscr{H}^{\downarrow}_\mathscr{R}$ (*white *for $x \to y$ and* red *for $x \to^* y$). Again, $[T]^2$ is* not *part of the graph. Also note the following:*

1. *Although the hook $\mathtt{d} \to^* x$ of $[Rl_{(2)}]^1$ and the (renamed) head $x' \to^* z'$ of $[T]^1$ unify with substitution $\theta = \{x' \mapsto \mathtt{d}, x \mapsto z'\}$, there is no arc from $[Rl_{(2)}]^1$ to $[T]^1$ because the goal $\theta(x' \to y') = \mathtt{d} \to y'$ (which is obtained by instantiation of the hook of $[T]^1$) does* not unify *with the head of any proof jump. Similarly, there is no arc from $[Rl_{(3)}]^1$ to $[T]^1$.*

[1] This can be proved automatically as explained in [9, 12] by using a tool like AGES or Mace4 [17].

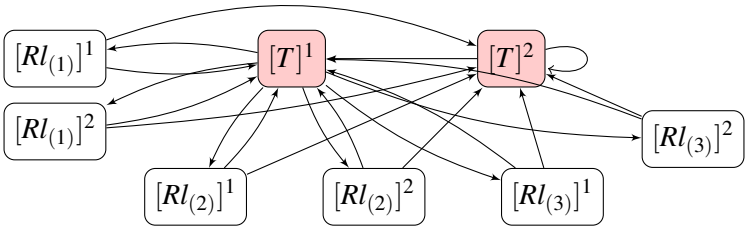
Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

2. *The hook* $\mathtt{e} \to^* x$ *of* $[Rl_{(2)}]^2$ *and the head* $x' \to^* z'$ *of* $[T]^1$ *unify with* $\theta = \{x' \mapsto \mathtt{e}, x \mapsto z'\}$, *but there is no arc from* $[Rl_{(2)}]^2$ *to* $[T]^1$ *because* $\theta(x' \to y') = \mathtt{e} \to y'$ *does* not unify *with the head of any proof jump. Similarly, there is no arc from* $[Rl_{(3)}]^2$ *to* $[T]^1$.

Remark 2 (Proof graph and directed proof graph) *If we let* $\mathscr{J} = \mathscr{J}_{\mathscr{S}}$ *and* $\Delta = \mathscr{H}_{\mathscr{S}}$ *in Definition 8, then we do* not *necessarily obtain the proof graph* $\mathsf{PG}(\mathscr{S}, \mathscr{J}_{\mathscr{S}})$ *in [14, Definition 9]: conditions (ii) and (iii) are missing in the definition of* $\mathsf{PG}(\mathscr{S}, \mathscr{J})$*; thus,* $\mathsf{DPG}(\mathscr{S}, \mathscr{J}, \Delta)$ *is (in general) a* subgraph *of* $\mathsf{PG}(\mathscr{S}, \mathscr{J})$.

As for the directed proof graph, the *proof graph* $\mathsf{PG}(\mathscr{S}, \mathscr{J}_{\mathscr{S}})$ which is mentioned in the previous remark is *not* computable and *estimations* $\mathsf{EPG}(\mathscr{S}, \mathscr{J}_{\mathscr{S}})$ are required to use it. Even in this case, we can see the advantages of considering estimations for $\mathsf{DPG}(\mathscr{S}, \mathscr{J}, \Delta)$ rather than for $\mathsf{PG}(\mathscr{S}, \mathscr{J}_{\mathscr{S}})$.

Example 14 *As shown in [14, Example 7], the estimated graph* $\mathsf{EPG}(\mathscr{S}, \mathscr{J}_{\mathscr{R}})$ *for the proof jumps in Example 4 is the following:*



For instance, there is an arc from $[T]^1$ *to* $[Rl_{(2)}]^2$ *because* $x \to y$ *and* $b \to d$ *unify with substitution* σ *such that* $\sigma(x) = b$ *and* $\sigma(y) = d$. *In contrast, there is no arc from* $[T]^2$ *to* $[Rl_{(2)}]^2$ *because* $y \to^* z$ *and* $b \to d$ *do* not *unify. This graph can be compared with the directed ones displayed in Example 13.*

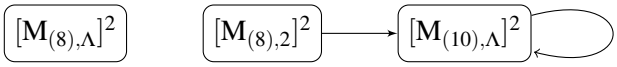
Example 15 *For the following TRS* $\mathscr{R}$ *[21, Example 21]:*

$$\mathtt{f}(x,\mathtt{a}) \to \mathtt{f}(x,\mathtt{g}(x,\mathtt{b})) \quad (8) \qquad \mathtt{g}(\mathtt{a},y) \to y \quad (9) \qquad \mathtt{g}(\mathtt{h}(x),y) \to \mathtt{g}(x,\mathtt{h}(y)) \quad (10)$$

and theory $\mathscr{R}_{\mathsf{m}}$ *obtained from* $\mathscr{I}_M(\mathscr{R})$ *(Figure 4),* $\mathscr{J}_{\mathscr{R}_{\mathsf{m}}}\downarrow_{\{\mathsf{m}(x)\}}$ *is as follows:*

$$[\mathrm{M}_{(8),\Lambda}]^2 \quad \mathsf{m}(X) \Uparrow X \xrightarrow{>\Lambda}{}^* \mathtt{f}(x,\mathtt{a}),\ \mathsf{m}(\mathtt{f}(x,\mathtt{g}(x,\mathtt{b}))) \qquad [\mathrm{M}_{(8),2}]^2 \quad \mathsf{m}(X) \Uparrow X \xrightarrow{>\Lambda}{}^* \mathtt{f}(x,\mathtt{a}),\ \mathsf{m}(\mathtt{g}(x,\mathtt{b}))$$
$$[\mathrm{M}_{(10),\Lambda}]^2 \quad \mathsf{m}(X) \Uparrow X \xrightarrow{>\Lambda}{}^* \mathtt{g}(\mathtt{h}(x),y),\ \mathsf{m}(\mathtt{g}(x,\mathtt{h}(y)))$$

The estimated $\{\mathsf{m}(x)\}$*-directed graph* $EDPG(\mathscr{R}_{\mathsf{m}}, \mathscr{J}_{\mathscr{R}_{\mathsf{m}}}\downarrow_{\{\mathsf{m}(x)\}}, \{\mathsf{m}(x)\})$ *is:*



The (absence of) arcs is justified as follows:

1. *No arc from* $[\mathrm{M}_{(8),\Lambda}]^2$ *to itself or to* $[\mathrm{M}_{(8),2}]^2$*:* $\mathsf{m}(\mathtt{f}(x,\mathtt{g}(x,\mathtt{b})))$ *and* $\mathsf{m}(X')$ *unify by* $\{X' \mapsto \mathtt{f}(x,\mathtt{g}(x,\mathtt{b}))\}$, *but* $\mathtt{f}(x,\mathtt{g}(x,\mathtt{b})) \xrightarrow{>\Lambda}{}^* \mathtt{f}(x',\mathtt{a})$ *is infeasible.*
2. *No arc from* $[\mathrm{M}_{(8),2}]^2$ *to itself or to* $[\mathrm{M}_{(8),\Lambda}]^2$*:* $\mathsf{m}(\mathtt{g}(x,\mathtt{b}))$ *and* $\mathsf{m}(X')$ *unify by* $\{X' \mapsto \mathtt{g}(x,\mathtt{b})\}$ *but* $\mathtt{g}(x,\mathtt{b}) \xrightarrow{>\Lambda}{}^* \mathtt{f}(x',\mathtt{a})$ *is infeasible.*
3. *No arc from* $[\mathrm{M}_{(10),\Lambda}]^2$ *to* $[\mathrm{M}_{(8),\Lambda}]^2$ *or* $[\mathrm{M}_{(8),2}]^2$*:* $\mathsf{m}(\mathtt{g}(x,\mathtt{h}(y)))$ *and* $\mathsf{m}(X')$ *unify by* $\{X' \mapsto \mathtt{g}(x,\mathtt{h}(y))\}$, *but* $\mathtt{g}(x,\mathtt{h}(y)) \xrightarrow{>\Lambda}{}^* \mathtt{f}(x',\mathtt{a})$ *is infeasible.*

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

6 Use of the OT-Framework for General Logics

In this section we show how the OT-Framework in [14] can be adapted to deal with directed OT-problems and then to prove Directed Operational Termination. An *OT problem* in $\mathscr{L}$ is a pair $\tau = (\mathscr{S}, \mathscr{J})$ with $\mathscr{S} \in Th(\mathscr{L})$ and $\mathscr{J} \subseteq Jumps(\mathscr{S})$; τ is *finite* if there is no infinite $(\mathscr{S}, \mathscr{J})$-chain; otherwise, τ is called *infinite*. The set of all OT problems in $\mathscr{L}$ is $OTP(\mathscr{L})$ [14].

Definition 10 *The* initial OT problem $\tau_I(\mathscr{S})$ *for a theory* $\mathscr{S}$ *is* $(\mathscr{S}^+, \mathscr{J}^+_{\mathscr{S}})$.

Example 16 *For* $\mathscr{R}$ *in Example 5,* $\tau_I(\mathscr{R}) = (\mathscr{I}(\mathscr{R}) - \{(Rl)_{(6)}\}, \mathscr{J}_{\mathscr{R}})$.

Theorem 9 [14] *A theory* $\mathscr{S}$ *is* operationally terminating *iff* $\tau_I(\mathscr{S})$ *is* finite.

We say that an OT problem $\tau = (\mathscr{S}, \mathscr{J})$ is Δ-directed (or a Δ-OT problem) iff $\mathscr{J} = \mathscr{J}\lfloor_\Delta$. Given a direction Δ, every OT problem $\tau = (\mathscr{S}, \mathscr{J})$ is transformed into a Δ-OT problem $\tau\lfloor_\Delta = (\mathscr{S}, \mathscr{J}\lfloor_\Delta)$.

Definition 11 *A* Δ*-OT problem* $\tau = (\mathscr{S}, \mathscr{J})$ *is* Δ-finite *iff there is no infinite* Δ*-directed* $(\mathscr{S}, \mathscr{J})$*-chain; otherwise,* τ *is called* Δ-infinite.

In view of Definitions 10 and 11, the following result just recasts Theorem 7.

Theorem 10 *A theory* $\mathscr{S}$ *is* Δ*-DOT iff* $\tau_I(\mathscr{S})\lfloor_\Delta$ *is* Δ*-finite.*

In the *OT Framework* for proving operational termination [14], an *OT processor* (or just *processor*) $\mathsf{P} : OTP(\mathscr{L}) \to \mathscr{P}(OTP(\mathscr{L})) \cup \{\mathsf{no}\}$ maps an OT problem into either a *set of OT problems* or the answer "no". A processor P is *sound* if for all OT problems τ, if $\mathsf{P}(\tau) \neq \mathsf{no}$ and all OT problems in $\mathsf{P}(\tau)$ are finite, then τ is finite. A processor P is *complete* if for all OT problems τ, if $\mathsf{P}(\tau) = \mathsf{no}$ or $\mathsf{P}(\tau)$ contains an infinite OT problem, then τ is infinite.

In the following, a processor P that maps a Δ-OT problem into either a *set of* Δ*-OT problems* or "no" is called a Δ-processor. The notions of Δ-soundness and Δ-completeness are obtained from the straightforward adaptation of the previous ones when Δ is considered. A Δ-OT problem τ is Δ-finite if it is finite (as an OT problem). If $\mathscr{I}(\mathscr{S})$ is finite, then every infinite OT-problem τ can be seen as a Δ-infinite Δ-OT problem $\tau\lfloor_\Delta$ for some Δ (which is obtained from the heads and hooks of the proof jumps in τ). Given a processor P and a direction Δ, we obtain a processor P_Δ that maps Δ-OT problems into sets of Δ-OT problems as follows:

$$\mathsf{P}^\Delta(\tau) = \begin{cases} \mathsf{no} & \text{if } \mathsf{P}(\tau) = \mathsf{no} \\ \{(\mathscr{S}, \mathscr{J}'\lfloor_\Delta) \mid (\mathscr{S}, \mathscr{J}') \in \mathsf{P}(\tau)\} - \{(\mathscr{S}, \emptyset)\} & \text{otherwise} \end{cases}$$

With this definition P^Δ is Δ-sound whenever P is *sound*. Depending on Δ, though, there can be infinite Δ-problems that are *not* Δ-infinite (there could be a strict subset Δ_0 of Δ subsuming proof jumps defining an infinite chain without requiring the participation of proof jumps subsumed by $\Delta - \Delta_0$). However, if Δ is a *singleton*, then P^Δ is Δ-complete whenever P is complete.

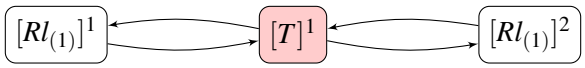
By repeatedly applying processors, we can construct a tree (called an *OT-tree*) for an OT-problem τ_I whose nodes are labeled with OT problems or "yes" or "no", and whose root is labeled with τ_I. For every inner node labeled with τ, there is a processor P satisfying one of the following: (i) $\mathsf{P}(\tau) = \mathsf{no}$ and the node has just one child that is labeled with "no". (ii) $\mathsf{P}(\tau) = \emptyset$ and the node has just one child that is labeled with "yes". (iii) $\mathsf{P}(\tau) \neq \mathsf{no}$, $\mathsf{P}(\tau) \neq \emptyset$, and the children of the node are labeled with the OT problems in $\mathsf{P}(\tau)$.

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018).
https://hdl.handle.net/11705/PROLE/2018/009

Theorem 11 (Directed OT-Framework) *Let τ be a Δ-OT problem. If all leaves of an OT-tree for τ are labeled with* "yes" *and all used processors are* Δ-sound*, then τ is Δ-finite. If there is a leaf labeled with* "no" *and all processors used on the path from the root to this leaf are* Δ-complete*, then τ is Δ-infinite.*

A *Strongly Connected Component* (SCC) in a graph is a *maximal* cycle, not contained in any other cycle. The SCC processor P_{SCC} [14, Section 5.2] just takes an OT problem τ, and returns a set with OT problems whose proof jumps are SCCs in the graph for τ.

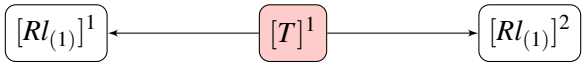
Example 17 *For $\mathscr{R}$ and $\mathscr{J}_{\mathscr{R}}$ in Example 4, the only SCC for* $\mathsf{EDPG}(\mathscr{R}, \mathscr{J}_{\mathscr{R}}\lfloor_{\{x\to y, x\to^* y\}}, \{x\to y, x\to^* y\})$ *in Example 13 is:*



Hence, $\mathsf{P}_{SCC}(\tau_I(\mathscr{R})\lfloor_{\{x\to y, x\to^* y\}}) = \{\tau'\}$*, for* $\tau' = (\mathscr{R}, \{[Rl_{(1)}]^1, [Rl_{(1)}]^2, [T]^1\})$*. Now,*

$$\mathsf{EDPG}(\mathscr{R}, \{[Rl_{(1)}]^1, [Rl_{(1)}]^2, [T]^1\}, \{x\to y, x\to^* y\})$$

is:



Indeed, although the hook $\mathsf{b}\to^* x$ *of* $[Rl_{(1)}]^1$ *and the (renamed) head* $x'\to^* z'$ *of* $[T]^1$ *unify with* $\theta = \{x'\mapsto \mathsf{b}, x\mapsto z'\}$*, there is no arc from* $[Rl_{(1)}]^1$ *to* $[T]^1$ *because the goal* $\theta(x'\to y') = \mathsf{b}\to y'$ *(which is the instantiation of the hook of* $[T]^1$*) does* not unify *with the head of any proof jump. Similarly, the hook* $\mathsf{e}\to^* x$ *of* $[Rl_{(1)}]^2$ *and the head* $x'\to^* z'$ *of* $[T]^1$ *unify with* $\theta = \{x'\mapsto \mathsf{c}, x\mapsto z'\}$*, but there is no arc from* $[Rl_{(2)}]^2$ *to* $[T]^1$ *because* $\theta(x'\to y') = \mathsf{c}\to y'$ *does* not unify *with the head of any proof jump. A second application of* P_{SCC} *yields* $\mathsf{P}_{SCC}^{\{x\to y, x\to^* y\}}(\tau') = \emptyset$*. By Theorem 11, this proves* $\{x\to y, x\to^* y\}$*-DOT of* $\mathscr{R}$*. Together with* $\{x\to^* y\}$*-DOT (see Example 13), by Theorem 5, we conclude operational termination of* $\mathscr{R}$*.*

In Example 18 below, we use the *removal pair* processor P_{RP} [14, Section 5.4]. We can remove proof jumps $(A \Uparrow \vec{B}_n)$ from OT problems $(\mathscr{S}, \mathscr{J})$ by using *removal pairs* $(\gtrsim, \sqsupset)$, where $\gtrsim$ and $\sqsupset$ are binary relations on *formulas* in $Form(\mathscr{S})$ such that $\sqsupset$ is *well-founded* and $\gtrsim$ is *compatible* with $\sqsupset$ (i.e., $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$) provided that the *hook* B_n is '*smaller*' (w.r.t. $\sqsupset$) than the *head* A.

Definition 12 [14] *Let $(\mathscr{S}, \mathscr{J}) \in OTP(\mathscr{L})$, $\psi : A \Uparrow \vec{B}_n \in \mathscr{J}$, and $(\gtrsim, \sqsupset)$ be a* removal pair*. Then, $\mathsf{P}_{RP}(\mathscr{S}, \mathscr{J}) = \{(\mathscr{S}, \mathscr{J} - \{\psi\})\}$ if and only if*

1. *for all $C \Uparrow \vec{D}_m \in \mathscr{J} - \{\psi\}$ and substitutions σ, if $\mathscr{S} \vdash \sigma(D_i)$ for all $1 \le i < m$, then $\sigma(C) \gtrsim \sigma(D_m)$ or $\sigma(C) \sqsupset \sigma(D_m)$, and*
2. *for all substitutions σ, if $\mathscr{S} \vdash \sigma(B_i)$* for all *$1 \le i < n$, then $\sigma(A) \sqsupset \sigma(B_n)$.*

Example 18 *For $\mathscr{R}$ in Example 5, we have $\tau_I(\mathscr{R}) = (\mathscr{I}(\mathscr{R}) - \{(Rl)_{(6)}\}, \mathscr{J}_{\mathscr{R}})$ (Example 16). By Theorem 4, we do not consider the direction $\{x\to y\}$. For direction $\{x\to y, x\to^* y\}$, we have $\mathscr{J}_{\mathscr{R}}\lfloor_{\{x\to y, x\to^* y\}} = \mathscr{J}_{\mathscr{R}}$, where the proof jumps in $\mathscr{J}_{\mathscr{R}}$ are $[C_{\mathtt{f},1}]^1$, $[C_{\mathtt{g},1}]^1$, $[T]^1$, $[T]^2$, and $[Rl_{(6)}]^1$. By using the methodology in [10, Section 5], we can prove that there is a well-founded relation $\sqsupset$* on formulas *such that, for all substitutions σ, all these proof jumps satisfy $\sigma(A) \sqsupset \sigma(B_n)$ provided that conditions $\sigma(B_1), \ldots, \sigma(B_{n-1})$, if any, are provable (which is important for $[T]^2$ only). An assisted proof can be obtained using* AGES *to automatically generate the relation $\sqsupset$. The following specification encodes the* feasible rules *of $\mathscr{R}$ (not including rule (6), because it leads to the infeasible inference rule $(Rl)_{(6)}$):*

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018).
https://hdl.handle.net/11705/PROLE/2018/009

```
mod ExGA01 is
  sort S T .
  ops a b : -> S .  ops f g : S -> S .  ops fto ftos : S S -> T .
  op wfr : T T -> Bool [wellfounded] .
  rl a => b .       rl f(a) => b .
endm
```

The required inference system is generated by AGES *according to Figure 1. The new functions* `fto` *and* `ftos` *(or $f_{\to}$ and $f_{\to^*}$ for short) are introduced to treat predicate symbols $\to$ and $\to^*$ as binary function symbols taking values in S and returning values in a new sort T on which relation* `wfr` *is defined; this relation represents the* well-founded relation $\sqsupset$. *In this way, formulas like $x \to^* z$ and $x \to y$ (the head and hook of $[T]^1$) are compared as* terms $f_{\to^*}(x,z)$ *and* $f_{\to}(x,y)$ *using* `wfr` *(see [10, Section 5]). The following formulas, representing the required* decreasingness *for the heads and hooks of all proof jumps:*

```
wfr(ftos(x:S,z:S),fto(x:S,y:S))
x:S -> y:S => wfr(ftos(x:S,z:S),ftos(y:S,z:S))
wfr(fto(f(x:S),f(y:S)),fto(x:S,y:S))
wfr(fto(g(x:S),g(y:S)),fto(x:S,y:S))
wfr(fto(g(x:S),g(a)),ftos(f(x:S),x:S))
```

are uploaded into AGES. *The following interpretation $\mathscr{A}$ computed by AGES permits a simultaneous removal of all proof jumps from $\tau_I(\mathscr{R})$: $\mathscr{A}_S = \mathscr{A}_T = \mathbb{N} \cup \{-1\}$; for function and predicate symbols:*

$$\begin{array}{rclrclrcl} \mathtt{a}^{\mathscr{A}} & = & 1 & \mathtt{b}^{\mathscr{A}} & = & 0 & \mathtt{f}^{\mathscr{A}}(x) & = & x+2 \\ \mathtt{g}^{\mathscr{A}}(x) & = & x+4 & f^{\mathscr{A}}_{\to}(x,y) & = & x & f^{\mathscr{A}}_{\to^*}(x,y) & = & x+1 \\ x \to^{\mathscr{A}} y & \Leftrightarrow & x > y \wedge y \geq 0 & x(\to^*)^{\mathscr{A}} y & \Leftrightarrow & 6x+2y+8 \geq 0 & x \sqsupset^{\mathscr{A}} y & \Leftrightarrow & x > y \end{array}$$

For $\Delta = \{x \to^* y\}$, the only proof jump in $\mathscr{J}_{\mathscr{R}}|_{\{x\to^* y\}}$ is $[T]^2$ and we proceed similarly (see Example 9). By Theorems 11 and 5, $\mathscr{R}$ is operational terminating.

Example 19 *Consider the following one rule TRS $\mathscr{R} = \{\mathtt{a} \to \mathtt{b}\}$, where* `a` *and* `b` *are constants. We prove termination of $\to^+_{\mathscr{R}}$ by using $\mathscr{I}_3(\mathscr{R})$ in Example 1 (which is* not *operationally terminating) together with Theorem 6. In this simple case, $\mathscr{I}_3(\mathscr{R})$ consists of* two *inference rules (due to the absence of nonconstant function symbols, no congruence rule $(C^+)_{f,i}$ is necessary):*

$$(Rl^+)\ \frac{}{\mathtt{a} \to^+ \mathtt{b}} \qquad (T^+)\ \frac{x \to^+ y \qquad y \to^+ z}{x \to^+ z}$$

According to Theorem 6, we have to prove $\{\mathsf{seq}_{\to^+}(x)\}$-DOT of $\mathscr{I}(\mathscr{R}) = \mathscr{I}_3(\mathscr{R}) \cup \{\rho_{\to^+}\}$, where $\rho_{\to^+}$ is $\frac{x\to^+ y \quad \mathsf{seq}_{\to^+}(y)}{\mathsf{seq}_{\to^+}(x)}$. The set of proof jumps is

$$\begin{array}{rcl} \mathscr{J}_{\mathscr{R}} & = & \{[T^+]^1, [T^+]^2, [\rho_{\to^+}]^1, [\rho_{\to^+}]^2\} \\ & = & \{x \to^+ z \Uparrow x \to^+ y,\ x \to^+ z \Uparrow x \to^+ y,\ y \to^+ z,\ \mathsf{seq}_{\to^+}(x) \Uparrow x \to^+ y, \\ & & \ \mathsf{seq}_{\to^+}(x) \Uparrow x \to^+ y,\ \mathsf{seq}_{\to^+}(y)\} \end{array}$$

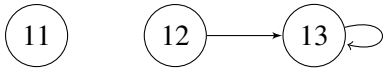
We have $\mathscr{J}_{\mathscr{R}}|_{\{\mathsf{seq}_{\to^+}(x)\}} = \{[\rho_{\to^+}]^2\}$. Therefore, in order to prove that $\mathscr{R}$ is $\{\mathsf{seq}_{\to^+}(x)\}$-DOT, we have to prove that $(\mathscr{R}, \{\mathsf{seq}_{\to^+}(x) \Uparrow x \to^+ y,\ \mathsf{seq}_{\to^+}(y)\})$ is $\{\mathsf{seq}_{\to^+}(x)\}$-finite. We can use P_{RP} for this purpose. We proceed as in the previous example to obtain a well-founded relation $\sqsupset$ on formulas such that $\sigma(\mathsf{seq}_{\to^+}(x)) \sqsupset \sigma(\mathsf{seq}_{\to^+}(y))$ for all substitutions σ such that $\overline{\mathscr{R}} \vdash \sigma(x) \to^+ \sigma(y)$ holds. In this case, the obtained model $\mathscr{A}$ has domains $\mathscr{A}_S = \{0,1\}$, $\mathscr{A}_T = \mathbb{N} \cup \{-1\}$, and interpretations for symbols as follows:

$$\mathtt{a}^{\mathscr{A}} = 1 \qquad \mathtt{b}^{\mathscr{A}} = 0 \qquad f^{\mathscr{A}}_{\mathsf{seq}_{\to^+}}(x) = x \qquad x(\to^+)^{\mathscr{A}} y \Leftrightarrow x > y \qquad x \sqsupset^{\mathscr{A}} y \Leftrightarrow 4x \geq 4y+1$$

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

7 Conclusions

For finite theories, the set of proof jumps is also finite and directions Δ capture all possible nonterminating behaviors as specific branches involving proof jumps described by Δ. This permits not only proving operational termination of the theory (as Δ-directed operational termination for all the finitely many directions Δ), but also single out specific (non)terminating behaviors that can be meaningful for the considered logic and theory. For instance, we can distinguish different 'dimensions' of operational termination of CTRSs as specific *directions* (Section 4.1). We can prove that a binary relation defined by an inference system is well-founded as a special case of DOT (Section 4.2). This is useful when the inference system in use is not operational terminating (by design) or when there are components which are intrinsically nonterminating (e.g., the equality predicate defined by $\mathscr{I}_{EL}(E)$ in Section 4.2.1, which defines a nonterminating relation for all equational theories E due to the reflexivity axiom $(E_{=})_{s,t}$) or not focused in the particular applications (e.g., TRS $\mathscr{R}_2$ when a proof of relative termination of $\mathscr{R}_1$ with respect to $\mathscr{R}_2$ is attempted, see Section 4.2.2). We can also approach the dependency pair framework for TRSs by means of a simple transformation of logics (Section 4.3). In [14, Section 8] we already discussed the relationship between dependency pairs and proof jumps. The discussion in Section 4.3 (and Example 15 regarding [21, Example 21]) provides further insight on these differences. For instance, the *dependency graph* estimated in [21, Example 21] by using dependency pairs and tree automata techniques for the approximation (see also [12, Example 13]) is the following:



where the nodes are the following *dependency pairs* $\mathsf{DP}(\mathscr{R})$ for $\mathscr{R}$:

$$\mathtt{F}(x,\mathtt{a}) \;\to\; \mathtt{F}(x,\mathtt{g}(x,\mathtt{b})) \qquad (11)$$

$$\mathtt{F}(x,\mathtt{a}) \;\to\; \mathtt{G}(x,\mathtt{b}) \qquad (12)$$

$$\mathtt{G}(\mathtt{h}(x),y) \;\to\; \mathtt{G}(x,\mathtt{h}(y)) \qquad (13)$$

where $\mathtt{F}$ and $\mathtt{G}$ are new symbols. This graph and $EDPG(\mathscr{R}_{\mathsf{m}}, \mathscr{J}_{\mathscr{R}_{\mathsf{m}}}\downarrow_{\{\mathsf{m}(x)\}}, \{\mathsf{m}(x)\})$ in Example 15 are isomorphic, but the 'contents' of the nodes and the way they are connected by arcs is different, although related due to Proposition 3.

With regard to the OT Framework [14], besides the obvious difference due to the use of *directions* (which is missing in [14]) the new approach developed in Sections 5 and 6 is also promising to prove operational termination, in view of the considered examples whose treatement is simpler. However, the combinatory effort of dealing with all possible directions (Theorems 2 and 3) should be further investigated to envisage appropriate simplifications. Another important issue which deserves further investigation is the connection between directed OT and *localized operational termination* [16] which has to do with considering the infinite proof trees obtained when trying to prove *specific goals* only.

References

[1] T. Arts and J. Giesl. Termination of Term Rewriting Using Dependency Pairs. *Theoretical Computer Science*, 236(1–2):133–178, 2000.

[2] F. Baader and T. Nipkow. Term Rewriting and All That. Cambridge University Press, 1998.

[3] A. Geser. Relative Termination. PhD Thesis. Fakultät für Mathematik und Informatik. Universität Passau, 1990.

[4] J. Giesl and T. Arts. Verification of Erlang Processes by Dependency Pairs. *Applicable Algebra in Engineering, Communication and Computing* 12:39-72, 2001.

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009

[5] J. Goguen and J. Meseguer. Models and Equality for Logical Programming. In *Proc. of TAPSOFT'87*, LNCS 250:1-22, 1987.

[6] R. Gutiérrez, S. Lucas, and P. Reinoso. A tool for the automatic generation of logical models of order-sorted first-order theories. In *Proc. of PROLE'16*, pages 215–230, 2016.

[7] N. Hirokawa and A. Middeldorp. Dependency Pairs Revisited. In V. van Oostrom, editor, *Proc. of RTA'04*, LNCS 3091:249–268, 2004.

[8] R. Lalement. Computation as Logic. Masson-Prentice Hall International, 1993.

[9] S. Lucas. Analysis of Rewriting-Based Systems as First-Order Theories. In *Revised Selected papers from LOPSTR 2017*, LNCS volume 10855, to appear, 2018.

[10] S. Lucas. Use of Logical Models for Proving Operational Termination in General Logics. In *Selected papers from WRLA'16*, LNCS 9942:1-21, 2016.

[11] S. Lucas and R. Gutiérrez. Automatic Synthesis of Logical Models for Order-Sorted First-Order Theories. *Journal of Automated Reasoning* 60(4):465–501, 2018.

[12] S. Lucas and R. Gutiérrez. Use of logical models for proving infeasibility in term rewriting. *Information Processing Letters*, 136C:90-95, 2018.

[13] S. Lucas, C. Marché, and J. Meseguer. Operational termination of conditional term rewriting systems. *Information Processing Letters*, 95:446–453, 2005.

[14] S. Lucas and J. Meseguer. Proving Operational Termination of Declarative Programs in General Logics. In *Proc. of PPDP'14*, pages 111-122, ACM 2014.

[15] S. Lucas and J. Meseguer. Dependency pairs for proving termination properties of conditional term rewriting systems. *Journal of Logical and Algebraic Methods in Programming*, 86:236-268, 2017.

[16] S. Lucas and J. Meseguer. Localized operational termination in general logics. In *Software, Services, and Systems* , LNCS 8950:91–114, 2015.

[17] W. McCune Prover9 & Mace4. `http://www.cs.unm.edu/~mccune/prover9/`, 2005–2010.

[18] J. Meseguer. Conditional rewriting logic as a unified model of concurrency. *Theoretical Computer Science* 96:73-155, 1992

[19] J. Meseguer. Membership algebra as a logical framework for equational specification. In *Proc. of WADT'97*, LNCS 1376:18–61, 1998.

[20] J. Meseguer. Twenty years of rewriting logic. *Journal of Logic and Algebraic Programming* 81:721-781, 2012.

[21] A. Middeldorp. Approximating Dependency Graphs Using Tree Automata Techniques. In *Proc. of IJCAR'01*, LNAI 2083:593–610, 2001.

[22] E. Ohlebusch. *Advanced Topics in Term Rewriting*. Springer-Verlag, 2002.

[23] G. Plotkin. A Note on Inductive Generalization. *Machine Intelligence* 5:153-163, 1970.

[24] R.M. Smullyan. Theory of Formal Systems. Princeton University Press, 1961.

Lucas, S.: Directions of Operational Termination. In: Ortega Mallén, Y. (ed.) Actas de las XVIII Jornadas de Programación y Lenguajes (PROLE 2018). Sistedes (2018). https://hdl.handle.net/11705/PROLE/2018/009